THE

MATHEMATICAL

GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

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VOL. XI. MARCH, 1923. No. 163.

The Mathematical Association.

President: SIR T. L. HEATH, K.C.B., K.C.V.O., D.Sc., F.R.S.

THE Annual Meeting of the Mathematical Association was held at the London Day Training College, Southampton Row, London, W.C. 1, on Monday, 1st January, 1923, at 5.30 p.m., and Tuesday, 2nd January, 1923, at 10 a.m. and 2.30 p.m.

MONDAY EVENING, 5.30 *p.m.*

Mr. S. Brodetsky, M.A., B.Sc., Ph.D., gave an address on "Gliding."

TUESDAY MORNING, 10 *a.m.*

BUSINESS.

(1) The following Report of the Council for the year 1922 was distributed and taken as read:

REPORT OF THE COUNCIL FOR THE YEAR 1922.

DURING the year 1922, 102 new members of the Association have been elected, and the number of members now on the Roll is 884. Of these, 8 are honorary members, 52 are life members by composition, 24 are life members under the old rule, and 800 are ordinary members. The number of associates connected with Local Branches of the Association is about 460.

The Council regret to have to record the deaths of Dr. Sophie Bryant; Mr. T. G. Creak, Honorary Secretary of the North Wales Branch; Mr. W. Finlayson, of Edinburgh; Professor W. Foord-Kelcey, of the Royal Military Academy; Dr. H. E. Girdlestone; the Rev. J. H. Kirkby, of Lechlade; Dr. C. G. Knott, Secretary of the Royal Society of Edinburgh; Major H. R. Ladell, J.P., of

North Walsham, formerly Headmaster of the London International College, Isleworth, and a member of the Association since 1877; Mr. F. E. Marshall, formerly of Harrow, and an original member of the Association for the Improvement of Geometrical Teaching, when it was founded in 1871; Dr. G. B. Mathews, F.R.S., President of the Association for the years 1905-7; Mr. H. G. Mayo, of Wallasey; Mr. A. Newham, Assistant Professor of Mathematics at Sydney University; and Dr. R. L. Woollcombe, of Dublin.

In the report of the Council for the year 1921, reference was made to the need of a new scheme for the election of the Teaching Committees. Such a scheme has been prepared, and has been distributed to all the members of the Association. It is now laid before the Annual Meeting for adoption, if approved.

Early in the year a Queensland branch of the Association was formed at Brisbane. Professor H. J. Priestley was elected President, and Mr. S. G. Brown, Honorary Secretary and Treasurer.

The Reports on the Teaching of Elementary Mathematics, 1902-1908, being sold out, the Council decided to reprint the Reports on the Teaching of Arithmetic and Algebra, Elementary Mechanics and Advanced School Mathematics. These can now be obtained from Messrs. G. Bell & Sons, under one cover, at the price of 1s. net.

The Council have appointed a Sub-Committee to report on the question of the library and its future. Suggestions from members of the Association would be welcomed.

Miss E. Glauert and Mr. J. Strachan now retire from the Council by rotation, and are not eligible for re-election for the coming year. The members present at the Annual Meeting will be asked to nominate and elect others to fill the vacancies.

The Council again desire to acknowledge the indebtedness of the Association to Mr. W. J. Greenstreet for his services as Editor of the *Mathematical Gazette.*

(2) The Treasurer briefly indicated the nature of the Report for the year 1922.

(3) The following Report of the General Committee of the Teaching Committee was then read.

The General Committee met once during the year and the Executive Committee twice.

The correspondence initiated in 1921 with the Oxford and Cambridge Local Syndicates and the Northern Universities Joint Board, with a view to securing uniformity in their mathematical syllabuses, was brought to a satisfactory conclusion; all three bodies modified their syllabuses by adopting the suggestions of this Association.

A special sub-committee has been appointed to consider speedily criticisms of examination questions. At their request the attention of the Registrars of the University of London was called to certain errors in recent examination papers.

The main feature of the year has been the appointment of a Committee to draw up a new report on the Teaching of Geometry; this Committee has not

yet completed its work, but a report of the work so far done is to be given in the course of the morning by Prof. Neville.

W. E. PATERSON, *Hon. Sec.*

(4) The Election of Officers and Members of Council for the year 1923 was then proceeded with.

Miss M. J. Griffith, Putney County Secondary School, and Mr. H. K. Marsden, Eton College, were elected Members of Council in place of Miss Glauert and Mr. J. Strachan, who retired by rotation.

(5) A new scheme, proposed by the Council, for the election of the Teaching Committee was presented to the Association.

(6) Prof. E. H. Neville, M.A., made a short statement *re* the forthcoming Report of the Sub-Committee on the Teaching of Geometry.

The following papers were read in the course of the meeting; the rest will appear in later numbers.

GLEANINGS FAR AND NEAR.

159. (Daniel Deronda enters the Jewish second-hand bookshop in Holborn to buy Salomon Maimon's *Lebensgeschichte*, and meets the Jew Mordecai.)

"In most other trades you find generous men who are anxious to sell you their wares for your own welfare: but even a Jew will not urge Simson's Euclid on you with an affectionate assurance that you will have pleasure in reading it, and that he wishes he had twenty more of the article, so much is it in request."—*Daniel Deronda*, Book IV., Chap. XXXIII.

["Simson" is of course the Glasgow professor of mathematics (1711-1761), Robert Simson of "Simson's Line" fame, on whose revision and extension of Euclid's *Elements* most subsequent editions have been based. The Euclid was published at Glasgow in 1756: *Daniel Deronda* appeared 120 years later.] —*Per* J. M'Whan.

160. Enigma. *From a Mathematical Treatise by Thomas Kersey.*

If the difference between the indices of the second letter of the second word and the third letter of the first word be multiplied into the differences of their squares, the product will be 576; and if their sum be multiplied into the sum of their squares, that product will be 2336, the index of the said third letter being the greatest. The indices of the last formed are the extremes of four numbers in arithmetical progression, the lesser mean being the index of the first letter of the third word; and the greater mean is the index of the fourth and last letter of the first word. The second letter of the third word is the same with the third letter of the first word, and the fifth letter of the third word is the same with the last letter of the first word. The sum of the squares of the indices of the first and second letters of the first word is 520, and the product of the same indices is seven-ninths of the square of the greater index, which is the index of the said first letter. The difference between the last two indices is the index of the first letter of the second word. The third and last letter of the second word, also the third letter of the third word, are the same with the second letter of the first word. The sum of the indices of the fourth letter of the third word, and the sixth or last letter of the same word, being added to their product, is 35; and the difference of their squares is 288, the index of the last letter being the least. Query—the words?

PRESIDENTIAL ADDRESS.

GREEK GEOMETRY WITH SPECIAL REFERENCE TO INFINITESIMALS.

By Sir T. L. Heath, K.C.B., K.C.V.O., D.Sc., F.R.S.

It may be convenient that, before approaching the special subject of my address, I should indicate very briefly the stages through which Greek geometry passed from its inception to the time of Archimedes and Apollonius, when it reached its highest development.

The story begins with Thales, who lived approximately from 624 to 547 B.C., and whose work therefore belongs roughly to the first half of the sixth century. Thales travelled in Egypt and (so we are told) brought geometry from thence into Greece. So far as we can judge from the available records, Egyptian geometry consisted almost entirely of practical rules for the mensuration of plane figures, such as squares, triangles, trapezia, and of the solid content of measures of corn, etc., of different shapes. They were also able to construct pyramids of a certain slope by means of a particular arithmetical ratio, which is in fact the cotangent of the angle of slope. They knew that a triangle with sides in the ratios of the numbers 3, 4, 5 is right-angled, and they used the fact as a means of drawing right angles. But we look in vain in Egyptian records for any general theorem or for any vestige of a proof of such. What is remarkable about Thales is not the few elementary propositions attributed to him, namely that a circle is bisected by any diameter, that the base angles of an isosceles triangle are equal, that if two straight lines cut one another the vertically opposite angles are respectively equal, that the triangles of Euclid I. 26 are congruent, and that the angle in a semicircle is a right angle; the vital fact is that Thales was the first person to think of *proving* such things, and thereby originating the idea of geometry as a science in and for itself. It was here that the Greek instinct for science emerged; the Greek was not satisfied with facts, he wanted to know the why and wherefore, and he was not content till he could get a *proof* or an explanation which commended itself to his reason. It was by this new point of view, this inspiration, that the Greeks came to create the science of mathematics and, with it, scientific method. Kant calls it nothing less than an intellectual revolution. A light, he says, broke on the first man who demonstrated the property of the isosceles triangle whether his name was Thales or what you will; since from that point onward the road that must be taken could no longer be missed, and the safe way of a science was struck and traced out for all time.

After Thales come Pythagoras and the Pythagoreans. Pythagoras lived from about 572 to 497 B.C., so that his work may be taken to belong to the second half of the sixth century, while his successors cover a large part of the fifth century. Of Pythagoras we are told that he transformed geometry into a subject of liberal education, investigating the principles of the science from the beginning. That is to say, he was the first to make geometry a connected system beginning with definitions and the other necessary preliminary assumptions. From the story that he bribed a promising pupil to learn geometry by offering him sixpence for each proposition that he mastered, we may infer that the subject was developed in a series of propositions. Pythagoras himself is credited with the theorem of Eucl. I. 47, with the discovery of a theory of proportion (numerical in character) and with the construction of the cosmic figures, the five regular solids. The construction of the dodecahedron involves that of a regular pentagon, and that again the division of a straight line in extreme and mean ratio, which is a particular case of the method known as the *application of areas*. The simplest case of this method is that of Euclid I. 44, 45, which corresponds to simple division in arithmetic.

The Pythagoreans extended the method to cases of application with a certain excess or defect as in Euclid VI. 27-29, which propositions amount to the geometrical solution of the most general form of quadratic equation, provided that it has real roots. Application pure and simple, and application with excess or defect, denoted respectively by the words παραβολή, ὑπερβολή and ἔλλειψις, is the form in which Apollonius expresses the fundamental properties of the three conics, and this is why he gave the three conics for the first time the names parabola, hyperbola and ellipse.

The Pythagoreans then devised and applied extensively the two powerful methods of proportion and application of areas. They also knew the properties of parallel lines, and proved the theorem that the three angles of a triangle are together equal to two right angles. In short, the Pythagoreans developed the bulk of the elementary parts of geometry corresponding to Books I., II., IV., VI., and probably III., of Euclid's *Elements*. They also discovered the incommensurable, at all events in the case of the diagonal of a square in relation to its side. This had the disconcerting result of revealing the defect in the Pythagorean theory of proportion, which applied only to commensurable magnitudes.

The geometry of which we have so far spoken belongs to the *Elements*. But, before the body of the *Elements* was complete, the Greeks had advanced beyond the *Elements*. By the second half of the fifth century B.C. they had investigated three famous problems in higher geometry, (1) the squaring of the circle, (2) the trisection of any angle, (3) the duplication of the cube. The great names belonging to this period are Hippias of Elis, Hippocrates of Chios, and Democritus.

Hippias of Elis invented a certain curve described by combining two uniform movements (one angular and the other rectilinear) taking the same time to complete. Hippias himself used his curve for the trisection of any angle or the division of it in any ratio; but it was afterwards employed by Dinostratus, a brother of Eudoxus's pupil Menaechmus, and by Nicomedes for squaring the circle, whence it got the name τετραγωνίζουσα, *quadratrix*.

Hippocrates of Chios is mentioned by Aristotle as an instance to prove that a man may be a distinguished geometer and, at the same time, a fool in the ordinary affairs of life. He occupies an important place both in elementary geometry and in relation to two of the higher problems above mentioned. He was, so far as is known, the first compiler of a book of Elements; and he was the first to prove the important theorem of Eucl. XII. 2, that circles are to one another as the squares on their diameters, from which he further deduced that similar segments of circles are to one another as the squares on their bases. These propositions were used by him in his tract on the squaring of *lunes*, which was intended to lead up to the squaring of the circle.

Hippocrates also attacked the problem of doubling the cube. He did not indeed solve it, but he reduced it to another, namely that of finding two mean proportionals in continued proportion between two given straight lines; and the problem was ever afterwards solved in that form.

Democritus wrote a large number of mathematical treatises, the titles only of which are preserved. From the title "On irrational lines and solids" we gather that he wrote on irrationals. As we shall see later, Democritus realised as fully as Zeno the difficulty connected with the continuous and the infinitesimal. Democritus, again, was the first to state that the volume of a pyramid or a cone is one third of that of the prism or cylinder respectively on the same base and of equal height, though as regards the cone at least he could not give a rigorous proof; this was reserved for Eudoxus.

We come now to the first half of the fourth century B.C., the time of Plato, and here the important names are Archytas, Theodorus of Cyrene, Theaetetus and Eudoxus.

Archytas was the first to solve the problem of the two mean proportionals; he used a wonderful construction in three dimensions, determining a certain point as the intersection of three surfaces, a cone, a half-cylinder and an anchor-ring or tore with inner diameter *nil*.

Theodorus, Plato's teacher in mathematics, extended the theory of the irrational by proving that $\sqrt{3}$, $\sqrt{5}$, ... $\sqrt{17}$ are all incommensurable with 1. Theodorus's proof was evidently not general; and it was reserved for Theaetetus to comprehend all such irrationals in one definition, and to prove the property generally as in Eucl. X. 9. Much of the content of the rest of Euclid's Book X. (dealing with compound irrationals), as of Book XIII. on the five regular solids, was also due to Theaetetus.

Eudoxus, an original genius second to none (unless it be Archimedes) in the history of our subject, made two discoveries of supreme importance for the further development of Greek geometry.

(1) As we have seen, the discovery of the incommensurable rendered inadequate the Pythagorean theory of proportion, which applied to commensurable magnitudes only. It would no doubt be possible, in most cases, to replace proofs depending on proportions by others; but this involved great inconvenience, and a slur was cast on geometry generally. The trouble was remedied once for all by Eudoxus's discovery of the great theory of proportion, applicable to commensurable and incommensurable magnitudes alike, which is expounded in Euclid's Book V. Well might Barrow say of this theory that "there is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established." The keystone of the structure is the definition of equal ratios (Eucl. V., Def. 5); and twenty-three centuries have not abated a jot from its value, as is plain from the facts that Weierstrass repeats it word for word as his definition of equal numbers, and it corresponds almost to the point of coincidence with the modern treatment of irrationals due to Dedekind.

(2) Eudoxus discovered the method of exhaustion for measuring curvilinear areas and solids, to which, with the extensions given to it by Archimedes, Greek geometry owes its greatest triumphs. The method is seen in operation in Euclid XII. 1-2, 3-7 Cor., 10, 16-18. Props. 3-7 Cor. and 10 prove the theorems about the volume of the pyramid and cone first stated by Democritus.

Menaechmus, a pupil of Eudoxus, was the discoverer of the conic sections, two of which, the parabola and the hyperbola, he used for solving the problem of the two mean proportionals. If $a : x = x : y = y : b$, then $x^2 = ay$, $y^2 = bx$ and $xy = ab$. These equations represent, in Cartesian co-ordinates, and with rectangular axes, the conics by the intersection of which two and two Menaechmus solved the problem; in the case of the rectangular hyperbola it was the asymptote-property which he used.

We pass to Euclid's times. A little older than Euclid, Autolycus of Pitane wrote two books, *On the Moving Sphere*, a work on Sphaeric for use in astronomy, and *On Risings and Settings*. The former work is the earliest Greek text-book which has reached us intact.

Euclid flourished about 300 B.C. or a little earlier. Besides his great work, the *Elements*, he wrote other books on both elementary and higher geometry, and on the other mathematical subjects known in his day, namely astronomy, optics and music. To elementary geometry belong the *Data*, and *On Divisions* (*of figures*); also the *Pseudaria*, now lost, which was a sort of guide to fallacies in geometrical reasoning. The treatises on higher geometry are all lost; they include (1) the *Conics*, which covered almost the same ground as the first three books of Apollonius's *Conics*; (2) the *Porisms*, in three books, the importance and difficulty of which can be inferred from Pappus's account of it and the lemmas he gives for use with it: (3) the *Surface-Loci*, for which also Pappus gives lemmas; one of these implies that Euclid assumed as

known the focus-directrix property of the three conics, which is absent from Apollonius's *Conics*.

In the period between Euclid and Archimedes comes Aristarchus of Samos, famous for having anticipated Copernicus. Accepting Heraclides's view that the earth rotates about its own axis, Aristarchus went further and put forward the hypothesis that the sun itself is at rest, and that the earth, as well as Mercury, Venus, and the other planets, revolve in circles about the sun.

One work of Aristarchus, *On the sizes and distances of the Sun and Moon*, is extant in Greek. Thoroughly classical in form, it lays down certain hypotheses, and then deduces therefrom, by rigorous geometry, limits to the sizes and distances of the sun and moon. Though the book contains no word of the heliocentric hypothesis, it is highly interesting in itself, because we here find geometry used for the first time with a *trigonometrical* purpose. In effect Aristarchus finds arithmetical limits to the values of certain trigonometrical ratios of small angles, namely,

$$\tfrac{1}{18} > \sin 3^\circ > \tfrac{1}{20}, \quad \tfrac{1}{45} > \sin 1^\circ > \tfrac{1}{60}, \quad 1 > \cos 1^\circ > \tfrac{89}{90}.$$

We now come to the third century B.C., which marks the culmination of Greek geometry in the works of Archimedes and Apollonius. Archimedes belongs to the second part of my subject. Of Apollonius it need only be said that he was called the "Great Geometer" out of admiration for the *Conics*, a treatise of the greatest originality and power. He produces his conics in the most general way from any oblique cone, and develops their properties with reference to axes which are in general oblique, namely a diameter and the tangent at its extremity, the principal axes only appearing as a particular case. The most remarkable portion of the work is perhaps Book V., in which Apollonius treats of normals to the conics as maximum and minimum straight lines drawn from different points to the curve; in particular he works out some intricate propositions, from which we can without difficulty deduce the Cartesian equations of the evolutes of the three conics respectively.

I can now turn to the second part of my task and endeavour to trace the evolution of the infinitesimal calculus in Greek geometry. The Pythagoreans, as we said, discovered the incommensurable. Closely connected with the idea of the incommensurable is the fact that mathematical magnitudes are divisible *ad infinitum*. Applying the principle to, say, a straight line, we have to consider to what we ultimately come, assuming (what is actually impossible) that we can carry the division to an end. Is it an indivisible line, or a point, or what? The Pythagoreans could not admit the possibility of an indivisible line: and, on the other hand, Pythagorean writers like Nicomachus are clear that you cannot make up a line out of points. What then is the ultimate product of the infinite division of a line, assuming it to be carried out? If it is neither an indivisible line nor a point, is it something which is different from both, in short, an infinitesimal? Zeno denied this, his argument having apparently taken this form. If magnitudes are divisible *ad infinitum*, the division must end, not in the infinitely small, because that is by hypothesis still divisible, but in *nothing*; if, however, you reverse the process and add nothing to itself for ever and ever, you will never produce any finite magnitude at all. The argument in the first of Zeno's paradoxes (the *Dichotomy*) is much the same. Before a moving thing can get any distance it must have travelled half that distance; before it has travelled the half it must have travelled the half of that, and so on, to the smallest distance; how then can it start at all? Democritus stated a similar dilemma about continuous variation in his question about consecutive sections of a cone parallel to the base. Suppose that we have a section of a cone parallel to its base and "indefinitely near" to it (as the phrase is), what are we to say of this section? Is it, said Democritus, equal, or not equal, to the base? If it is equal, so will the next consecutive section be, and so on; thus the cone will really be not a cone at all but a

cylinder. If, on the other hand, it is not equal to the base, and in fact less, the surface of the cone will be jagged, like steps, which is very absurd.

There is no doubt that the paradoxes of Zeno, and the difficulty of answering them, profoundly affected the formal development of mathematics from his time onward. Antiphon the Sophist, a contemporary of Socrates, had asserted, in connexion with attempts to square the circle, that if in a circle we inscribe successive regular polygons, beginning from a triangle or a square and continually doubling the number of sides, we shall sometime arrive at a polygon the sides of which will coincide with the circumference of the circle. Warned by the unanswerable arguments of Zeno, mathematicians henceforth substituted for this the statement that, by continuing the construction, we can inscribe a polygon approaching equality with the circle *as nearly as we please.* The method of exhaustion used, for the purpose of proof by *reductio ad absurdum,* the lemma proved in Eucl. X. 1 (to the effect that, if from any magnitude we subtract not less than its half, then from the remainder not less than half, and so on continually, there will sometime be left a magnitude less than any assigned magnitude of the same kind, however small); and this again depends on an assumption which is practically contained in Eucl. V. Def. 4, but is generally known as the Axiom of Archimedes, stating that, if we have two unequal magnitudes, their difference (however small) can, if continually added to itself, be made to exceed any given magnitude of the same kind (however great).

It was Eudoxus who proved by the method of exhaustion the propositions about the volumes of a cone and a pyramid; and it is probably to Eudoxus that Euclid owed the proofs of the theorems that circles are to one another as the squares on their diameters, and that spheres are to one another as the cubes on their diameters.

The notion of *exhausting* the circle was, however, that of Antiphon, and he is entitled to credit for an idea which proved so fruitful. Geometers continued to use the method of exhausting an area indicated by Antiphon, while they barred themselves, in theory at least, from carrying the process to the point of taking a limit. They would certainly have denied that an inference of this kind could constitute a valid proof. There is a passage in the recently discovered tract of Archimedes, the *Method,* which bears this out. The only form of proof which was considered to be conclusive was the double *reductio ad absurdum* which was part of the method of exhaustion.

Some remarks about this method of proof are necessary at this stage. The process will be familiar to you from Euclid XII. 2. That proposition and those about the volume of the cone and pyramid suggest that Eudoxus approximated to the figure to be measured by inscribed figures only, the result of which is that the second part of the proof has to be inverted. Archimedes avoided this by approximating both from above and from below, so to speak, *i.e.* by circumscribing figures as well as inscribing them, so that both parts of the proof are direct.

Writers have sometimes observed that, before we can apply the method of exhaustion, we must already know the result to be proved, implying that the method is of no use for discovering such results. This is scarcely true, because the method of exhaustion does not consist exclusively of the proof by *reductio ad absurdum*; it includes the process of exhausting the area or volume to be measured, and this in itself often indicates the result quite clearly.

But, in view of the criticism, it is desirable to consider exactly how the Greeks, and Archimedes in particular, actually obtained the various results which were then to be established by the rigorous proof. We can distinguish three classes of cases.

In the first the result was divined in a way which, however careful the Greeks were to avoid mentioning such a thing, amounted to passing to the limit. For example, there can be no doubt that the theorem that circles are to one another

as the squares on their diameters was inferred on Antiphon's lines: since similar inscribed polygons are in the ratio in question, the same must be true of similar regular polygons inscribed in the circles when the number of sides is indefinitely increased, and their length correspondingly diminished, and therefore of the circles themselves from which the polygons will then be indistinguishable. Another case is the proposition in Archimedes' *Measurement of a Circle*, to the effect that the area of a circle is equal to a triangle with height equal to the radius and base equal to the circumference of the circle; the very form of this enunciation suggests that it was arrived at by regarding the circle as the sum of an indefinitely large number of isosceles triangles with the centre as common vertex and with equal indefinitely small chords of the circle as bases. Again, when Democritus concluded that the volume of a cone is one third of that of the cylinder with the same base and height, he must have inferred this from the consideration that, the corresponding proposition being true of a pyramid with a polygon of any number of sides as base, it is still true when the number of sides of a regular polygon forming the base is indefinitely increased, and the sides correspondingly diminished, so that the polygon tends to become indistinguishable from a circle and the pyramid from a cone. But perhaps the clearest case of all is in the *Method* of Archimedes, where he says that from the volume of a sphere (already known to be what we write as $\frac{4}{3}\pi\rho^3$) he inferred that the surface of the sphere is $4\pi\rho^2$; his words are, "Starting from the fact that any circle is equal to a triangle the base of which is the circumference, and the height the radius, of the circle, it occurred to me that, just in the same way, any sphere is equal to a cone the base of which is the surface, and the height the radius, of the sphere."

The second class of case is that in which the process of exhausting the area or content itself indicates the result. In this class of case what happens is that we get a series of terms that can be summed; and there are two varieties according to the nature of the series that has to be summed. One case is simple, depending on the summation of the geometrical progression

$$1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\ldots \textit{ad inf.}$$

This is all that is required in Archimedes' second method of finding the area of any parabolic segment. He inscribes in the segment a triangle with the same base and vertex, then in the two segments left over two triangles with the same vertex and base respectively, in the four segments left over four more triangles, and so on. If A is the area of the first triangle, the sum of the next two triangles is $\frac{1}{4}A$, the sum of the next four triangles $(\frac{1}{4})^2A$, and so on. Adding the series of triangles, we have

$$A\{1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\ldots\}.$$

Archimedes sums n terms of the series, stating the result in the form

$$A\{1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\ldots+(\tfrac{1}{4})^{n-1}\}+\frac{A}{3}(\tfrac{1}{4})^{n-1}=\tfrac{4}{3}A.$$

To find the sum of the series continued *ad infinitum* we should simply observe that $(\frac{1}{4})^{n-1}$ can be neglected when n is indefinitely increased, so that the sum of the series to infinity is $\frac{4}{3}$ and the area of the segment is $\frac{4}{3}A$. Although Archimedes probably inferred the result in this way, he does not say so, but simply declares that the area is $\frac{4}{3}A$, and then proves it by *reductio ad absurdum* with the help of the summation to n terms only.

The second kind of series the summation of which gives the desired result is more important, the series taking such forms that their summation is really equivalent to an *integration*. In the works of Archimedes there are some six cases where the investigation gives the equivalent of an integration. Three of the actual integrals are $\int x\,dx$, $\int x^2dx$ and a combination of the two. The volume of any segment of a paraboloid of revolution is obtained in a form

equivalent to $C\int_0^c x\,dx$, where C is a constant. In three cases (those of the volume of half a spheroid, the area cut off by a spiral between two radii vectores or one radius vector and the initial line, and the first investigation of the area of a parabolic segment) the procedure is equivalent to finding the integral $\int_0^b x^2dx$, and in the case of the spiral the integral $\int_b^c x^2dx$ also. The investigation of the volume of any segment of a hyperboloid of revolution amounts to finding the integral $\int_0^b (ax+x^2)dx$, which, by means of a certain device, is made to serve for finding the volume of any segment of a spheroid also.

I will illustrate by the simplest case, the paraboloid of revolution. Taking an oblique segment of it and a section of the segment by a plane through the axis of revolution, we have a parabola referred in general to oblique axes, PV and PE the tangent at its extremity. Let the equation of the parabola be $y^2=px$. PV being the axis of the paraboloidal segment to be measured, let PV be divided into n equal small parts of length h. Figures are inscribed and circumscribed to the segment made up of short oblique cylinders as indicated in the figure. To avoid taking account of the constant due to the

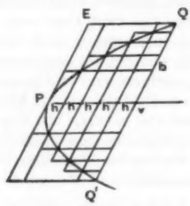

FIG. 1.

inclination of the axes, Archimedes compares each small cylinder with the corresponding portion of the whole cylinder EQ'.

If y is the ordinate corresponding to the abscissa rh, then the small cylinder having as base the section of the paraboloid through y parallel to the tangent plane at P is to the corresponding portion of the cylinder EQ' as y^2 to QV^2, while $y^2=p\,.\,rh$ and $QV^2=p\,.\,nh$. We have then

$$\frac{\text{(circumscribed figure)}}{\text{cylinder } EQ'}=\frac{p(h+2h+\ldots+nh)h}{p\,.\,nh\,.\,nh}$$

and

$$\frac{\text{(inscribed figure)}}{\text{cylinder } EQ'}=\frac{p(h+2h+\ldots+\overline{n-1}\,h)h}{p\,.\,nh\,.\,nh};$$

and Archimedes simply says that the first of the two ratios is $>$ and the second $<\frac{1}{2}$.

Now, if we sum the series, the ratios work out to

$$\frac{1}{n^2}\cdot\tfrac{1}{2}n(n+1) \quad\text{or}\quad \tfrac{1}{2}\frac{n+1}{n}$$

and

$$\frac{1}{n^2}\cdot\tfrac{1}{2}n(n-1) \quad\text{or}\quad \tfrac{1}{2}\frac{n-1}{n}.$$

When, therefore, Archimedes declares that the volume of the segment is one-half that of the cylinder, he is practically taking the limit of the summation when h is indefinitely small and n indefinitely large, but so that $nh=c$, and saying that

$$\text{Limit of } h(h+2h+\ldots+nh)=\tfrac{1}{2}c^2, \quad\text{or}\quad \int_0^c x\,dx=\tfrac{1}{2}c^2.$$

The corresponding inequality which Archimedes gives in the case of the area cut off by the spiral $r=a\theta$ between the initial line and the radius vector to any point on the first turn of the spiral works out to

$$h^2+(2h)^2+\ldots+(nh)^2>\tfrac{1}{3}n(nh)^2>h^2+(2h)^2+\ldots+\{(n-1)h\}^2.$$

He has in Prop. 10, *On Spirals*, summed the series

$$1^2+2^2+\ldots+n^2,$$

his result being equivalent to $\dfrac{n(n+1)(2n+1)}{6}$, from which the above inequalities readily follow.

If P be any point on the first turn, r the radius vector OP bounding the required area, θ the angle through which the generating line has turned from the initial line to OP, we suppose r to be divided into n parts h and draw the radii vectores h, $2h$, $3h$, ... $\overline{n-1}\,h$, which of course divide θ into equal parts θ/n. Drawing circular arcs with O as centre through the extremities of h, $2h$, $3h$ ... $(n-1)h$ and nh (OP), we have figures circumscribed and inscribed to the spiral which are made up of sectors of circles with vertical angle θ/n and radii h, $2h$, $3h$, ...

The area of the circumscribed figure is

$$\frac{\theta}{2n}\{h^2+(2h)^2+\ldots+(nh)^2\} \quad \text{or} \quad \frac{\theta}{2nh}\cdot h\{h^2+(2h)^2+\ldots+(nh)^2\},$$

and that of the inscribed figure is

$$\frac{\theta}{2nh}\cdot h\{h^2+(2h)^2+\ldots+(\overline{n-1}\,h)^2\}.$$

Archimedes then practically takes the limit of $h^3.\dfrac{n(n+1)(2n+1)}{6}$ when h is indefinitely small and n indefinitely large, while $nh=a\theta$, to be $\frac{1}{3}(nh)^3$, and the resulting area is

$$\frac{\theta}{2nh}\cdot\int_0^{nh}x^2dx \quad \text{or} \quad \frac{\theta}{2a\theta}\cdot\frac{1}{3}(a\theta)^3=\frac{\theta}{2}\cdot\frac{1}{3}(a\theta)^2.$$

In the case of a sphere and a segment of a sphere Archimedes does not proceed, as in the case of the spheroid, by means of one of the integrals above mentioned, probably because he has to find the surface as well as the volume of the sphere and segment, and to find the surface requires a different integration. In the case of the sphere and segment, therefore, he inscribes and circumscribes polygons to the generating circle and uses the figures generated by the revolution of these polygons about the axis, which figures are of course

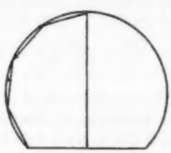

FIG. 2.

inscribed and circumscribed to the sphere or segment. The inscribed polygon is obtained by dividing the circumference of the section of the sphere or segment into $2n$ equal parts and joining the successive points of division; the circumscribed polygon has its sides parallel to those of the inscribed polygon. Each of the equal sides of the polygons subtends at the centre an angle α/n in the case of the segment (where 2α is the angle subtended by the arc at the centre), and π/n in the case of the whole circle.

To take the case of the segment, Archimedes proves by geometry the equivalent of the formula

$$\frac{2\left\{\sin\frac{a}{n}+\sin\frac{2a}{n}+\ldots+\sin(n-1)\frac{a}{n}\right\}+\sin a}{1-\cos a}=\cot\frac{a}{2n},$$

and deduces that the surface of the inscribed figure is

$$\pi a^2 . 2\sin\frac{a}{2n}\left[2\left\{\sin\frac{a}{n}+\sin\frac{2a}{n}+\ldots+\sin(n-1)\frac{a}{n}\right\}+\sin a\right],$$

that is, $$\pi a^2 . 2\cos\frac{a}{2n}(1-\cos a).$$

The effect of summing the series within brackets and taking the limit, which is practically what Archimedes does, is to show that

$$\pi a^2\int_0^a 2\sin\theta\, d\theta=2\pi a^2(1-\cos a).$$

Enough has been said to show the remarkable power in the hands of an Archimedes of the method of exhaustion even with the purely geometrical means available. The disadvantages of it were that it had, so far as the proof by *reductio ad absurdum* is concerned, to be applied separately to each case, and that there were only a limited number of real integrations which pure geometry could compass. The Greeks were thus limited because they had not made the discovery that differentiation and integration are the inverse of one another (a fact which was first fully proved by Barrow), and they were without the advantage, from the point of view of both, arising from algebraical notation and the modern discovery of the development of various functions in the form of series.

Of considerations corresponding to the differential calculus there are very few traces in Greek geometry. One case, however, seems certain, though nothing of the kind is expressed in the text. There can be no doubt that

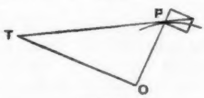

FIG. 3.

Archimedes obtained the sub-tangent property of the spiral $r=a\theta$ by a consideration of the instantaneous direction of the motion of the point describing the spiral at any point P of the curve. With our notation we have, by similar triangles,

$$\frac{OT}{r}=\frac{r\,d\theta}{dr}=\frac{r}{a},$$

and $OT=r^2/a$ or $r\theta$, which is the equivalent of Archimedes's statement of the property. We know next to nothing of the content of Apollonius's lost treatise on the *cochlias* or cylindrical helix; but if, as is probable, he gave the properties of the tangent to the curve at any point, he would no doubt similarly determine the direction of the tangent as the instantaneous direction of the motion of the describing point.

To return to cases normally demanding integration or its equivalent. The Greeks were by no means limited to what they could accomplish by direct integration. An alternative to direct integration was the reduction of the actual problem to another integration, the result of which was already known. A large proportion of the propositions in the *Method* of Archimedes had this object. But there are other cases on record where this must have been

the means of solution. Dionysodorus, we are told, found the volume of the tore or anchor-ring described by the revolution of a circle of radius a about a straight line in its plane at a distance c from the centre of the circle ($c>a$). Dionysodorus stated the result in this form:

$$\frac{\text{circle } BDCE}{\tfrac{1}{2}(\text{parallelogram } DH)} = \frac{(\text{volume of tore})}{(\text{cylinder rad. } EH \text{ and height } GH)}.$$

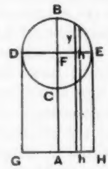

FIG. 4.

This gives $\quad (\text{volume of tore}) = \dfrac{\pi a^2 . 2\pi c^2 a}{ca}$

$$= 2\pi c . \pi a^2.$$

We can imagine how Dionysodorus arrived at this result. Dividing FE into equal small parts h and taking the corresponding ordinates y of the points on the circle $BDCE$, we find that the section of the tore formed by the revolution of the double ordinate $2y$ about the axis GH is $\pi(c+y)^2 - \pi(c-y)^2$ or πcy. We have therefore to find

$$\Sigma 2\pi cyh \quad \text{or} \quad 2\pi c . \Sigma h\sqrt{\{a^2-(rh)^2\}}$$

within the proper limits when h is indefinitely diminished. It would be obvious that $\Sigma h\sqrt{\{a^2-(rh)^2\}}$ is then the area of the circle $BCDE$, which is known, so that there is no need for an integration. The result is therefore, as stated, $2\pi c . \pi a^2$.

The cubature of the tore is otherwise historically interesting. Kepler, about 1615, solved it by means of infinitesimal sections. Any plane through the axis of revolution cuts the tore in two circular sections. Hence, said Kepler, we may regard the tore as made up of an infinite number of very thin discs. These discs are thinnest on the side towards the centre of the tore and thickest on the outside, and the mean thickness is the thickness at the centre of the discs. Adding the discs, we find that the volume of the tore is the same as if the circle passing through the centres of all the circular sections were straightened out and the tore turned into a cylinder of the same section. That is, the volume is $2\pi c . \pi a^2$. Professor Loria has remarked that Kepler's conception was bold and original. But the Greeks had in effect anticipated Kepler therein; for Heron of Alexandria, after mentioning Dionysodorus's solution, adds that the measurement can also be done in another way "if the tore is straightened out and so becomes a cylinder," and works out the figures on this basis.

The case of the tore investigated by Dionysodorus throws, I think, some light on the evolution of the more general proposition, claimed by Pappus as his own, which is in effect an anticipation of the theorem attributed to Guldin, that the volume of the solid formed by the revolution of any plane figure about an axis in its plane is the product of the area of the figure and the length of the path described by its centre of gravity. Divide the figure into narrow strips all of breadth h and all perpendicular to the axis of revolution.

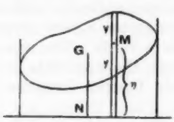

FIG. 5.

Let M be the centre of gravity of the strip of length $2y$, and η its distance from the axis of revolution. Then $2yh$ by its revolution about the axis produces a solid flat ring of content $h . \pi\{(\eta+y)^2-(\eta-y)^2\} = 2\pi . 2yh\eta$. Now

$\Sigma 2yh\eta$, when h is indefinitely diminished, is what we call the *moment* of the figure about the axis. What was wanted therefore to establish Pappus's theorem was simply the proposition that the moment of the whole figure placed where it is is equal to that of the whole figure supposed concentrated at its centre of gravity. The Greeks of course did not speak of "moments." They would have supposed the perpendicular GN from the centre of gravity G of the whole figure to the axis to be produced to H, so that $GN = NH$, and would have said that the figure placed where it is would balance about N a like figure supposed concentrated and placed at the point H. This is the phraseology of Archimedes's *Method*. Now we find that in the *Method* it is tacitly assumed, for each figure that is there treated, that when a figure is balanced where it is about a certain fulcrum against another figure on the other side, the former figure may be replaced by an equal weight placed at its centre of gravity. Archimedes in fact uses this proposition for the purpose of finding the centre of gravity of certain figures in a way corresponding to

$$\bar{\eta} \,.\, \mathrm{Lt}\, \Sigma 2yh = \mathrm{Lt}\, \Sigma 2y\eta h \quad \text{or} \quad \bar{\eta} \,.\int 2y\,dx = \int 2y\eta\,dx.$$

The theorem must therefore have been generally accepted at the time. No doubt it would be proved by some sort of integration.

I may perhaps suitably conclude with some remarks about Archimedes's treatise, the *Method*. I have mentioned that it illustrates the ingenuity with which Archimedes here obtains the results of certain integrations, not directly, but by reduction to other integrations the results of which are already known. But this treatise, found at Constantinople in 1906, has the supreme interest that it shows us for the first time how Archimedes originally discovered some of his results, namely by a peculiar method of his own depending on mechanical considerations. It is also a most important document in the history of infinitesimals, for here we find infinitesimals used almost without disguise.

The method is to weigh elements of the figure to be measured (X) against those of another (B) in such a way that the elements of B act at different points, namely where they are, but the elements of X act at one point. The latter point is a point on the diameter or axis of the figure to be measured produced in the direction away from the figure. The diameter or axis with the produced portion is imagined to be the bar or lever of a balance; the point of suspension is the extremity of the diameter or axis of the figure X. The content of B, as well as its centre of gravity, being known, its weight can be supposed to act as one mass at its centre of gravity. The corresponding elements of B and X, which are weighed against each other, are sections of B and X respectively by straight lines or planes parallel to the tangent line or plane at the extremity of the diameter or axis.

But the interesting fact is that the elements are spoken of as *straight lines* in the case of plane figures and *plane areas* in the case of solids. Herein Archimedes anticipated Cavalieri, who similarly speaks of figures being made up of linear or plane sections respectively, although in fact the elements are in the first case indefinitely narrow strips (areas), and in the second case indefinitely thin plane laminae (solids). The essential thing, as with Cavalieri, is that, though the number of the elements in each figure is infinite, the number is the same in both figures because the figures have the same height.

I will illustrate by one simple case, that of a right segment of a paraboloid of revolution.

BAC is a section through the axis of the paraboloid, AD the axis, ABC a section by the same plane of the cone with A as vertex and the circular section through BC as base, $EBCF$ the section by the plane through BAC of a cylinder on the same base as the cone.

PNQ is any double ordinate in the parabola *BAC*, and *PQ* produced both ways meets *EB*, *FC* in *L*, *M*.

Produce the axis *DA* to *H*, making *HA* equal to *DA*, and imagine *HAD* to be the bar of a balance, *A* being the point of suspension.

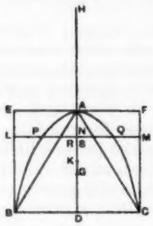

FIG. 6.

Now, by the property of the parabola,

$$DA : AN = BD^2 : PN^2,$$

or

$$\begin{aligned} HA : AN &= LN^2 : PN^2 \\ &= (\text{circle of rad. } LN) : (\text{circle of rad. } PN) \\ &= (\text{circle in cylinder}) : (\text{circle in paraboloid}). \end{aligned}$$

Consequently the circle in the paraboloid, if placed with its centre of gravity at *H*, balances the circle in the cylinder placed where it is and acting at *N*. Similarly for the respective circles in the two figures made by *all* plane sections at right angles to *AD*. Therefore the paraboloid acting as one mass at *H* balances the cylinder placed where it is, which again can be supposed to act as one mass concentrated at its centre of gravity, *i.e.* at *K* where *K* bisects *AD*.

Since then $\quad (\text{cylinder}) : (\text{paraboloid}) = HA : AK$

$$= 2 : 1,$$

the content of the paraboloid is half that of the cylinder and $\frac{3}{2}$ that of the cone *ABC*.

In the next proposition Archimedes uses the same procedure for finding the centre of gravity of the paraboloidal segment. This time he weighs the paraboloid where it is against the cone *ABC* supposed concentrated at *H*, thus:

Since $\quad PN^2 : BD^2 = AN : AD,$

and $\quad BD^2 : RN^2 = DA^2 : AN^2,$

it follows that $\quad PN^2 : RN^2 = DA : AN = HA : AN.$

Now *PN* is a radius of the section of the paraboloid, and *RN* a radius of the section of the cone *ABC*, by the plane section through *LM* and perpendicular to *AD*. Therefore the circle in the paraboloid where it is balances the circle in the cone when placed with its centre of gravity at *H*. So for the sections by all other planes at right angles to *AD*.

Therefore the paraboloid where it is balances the cone *ABC* supposed concentrated at *H*.

If now *G* is the centre of gravity of the paraboloid, we may suppose its weight to act at *G*.

Therefore $\quad (\text{paraboloid}) : (\text{cone}) = HA : AG.$

And the paraboloid is $1\frac{1}{2}$ times the cone; therefore

$$HA = \tfrac{3}{2}AG,$$

or

$$AG = \tfrac{2}{3}AD.$$

A METHOD OF STUDYING NON-EUCLIDEAN GEOMETRY.

BY W. C. FLETCHER, M.A.

IN the address which he had the honour of giving at the Annual Meeting of the Association the object of the writer was twofold: (1) to urge Secondary School teachers to study non-Euclidean Geometry for the purpose of improving their own scholarship; and (2) to sketch a method of approach to the subject which seems both interesting in itself and much easier than that usually adopted. In preparing this paper for the *Gazette* he has thought it better to omit the former and to develop the latter a little more fully than time allowed at the meeting; the title has been modified accordingly.

Euclidean Geometry like Mechanics grew up in the first instance on a basis of empirical knowledge; observation, measurement, direct intuition all played their part. Only later was it reduced to axioms, and only within the last century has the ultimate axiomatic basis been fully explored.

Non-Euclidean Geometry, on the other hand, was evolved strictly from axioms—not from the fully developed system of our days it is true, but that does not affect the main point—with little assistance from intuition: the discoverers did not know what the things were whose properties they were exploring, except in so far as their nature was embodied in the axioms.

This makes the subject essentially difficult; we are constantly under the impression of not knowing what it is all about.

While, of course, it is essential that the perfected science should be based on or reduced to axioms, it is not in the least essential for an individual student to commence his study of the subject with the axioms, or to follow the classical and strictly scientific path if an easier can be found. The object of this paper is to open such a path; it will enable any one more easily to get a sound knowledge of the subject matter; whether or not he thereafter examines the foundations should depend on his own tastes and purposes.

POINCARÉ'S IMAGINARY UNIVERSE.

Imagine, says Poincaré,* a world enclosed in a vast sphere of radius k, and subject to the following laws: (1) the absolute temperature at a point distant r from the centre is k^2-r^2; (2) all bodies whatsoever have the same coefficient of expansion, and their *linear* dimensions are proportional to the temperature; (3) bodies have no specific heat and take the temperature proper to their position instantaneously; (4) the index of refraction is

$$1/(k^2-r^2).$$

In this universe the line which the inhabitants would regard as "straight" would be that which we (the external observers) call an arc of a circle cutting the boundary at right angles; for three reasons:

(1) this would be the path of a ray of light;

(2) it would be the shortest track from point to point as measured by their instruments;

(3) it would be the axis of rotation, *i.e.* the locus of the points at rest in a body fixed at two points and revolving.

It is easy to verify, by forming and solving the necessary differential equation, that the path of a ray of light is an orthogonal arc as stated. It follows that along such a line $\int \mu \, ds$ is a minimum, *i.e.* that $\int \frac{ds}{k^2-r^2}$ is a minimum.

* *La Science et l'Hypothèse, passim.*

But the laws have been so chosen that $\frac{ds}{k^2 - r^2}$, or, for homogeneity and convenience, $\frac{2k^2 ds}{k^2 - r^2}$ is the length of the element ds according to the inhabitants' measurement. Call this $d\sigma$; then $\int_A^B d\sigma$ is a minimum along the path of the ray, *i.e.* along an orthogonal arc. [This can, of course, be verified directly by use of the Calculus of Variations.]

That an axis of rotation is also of the same form will probably be harder to realise at first, but it will become clear as we proceed.

We will then take as our new "straight lines" these orthogonal arcs; it easily follows that the new "planes" will be segments of spheres, also orthogonal to the boundary. These "straight lines" satisfy the axioms of hyperbolic geometry, *i.e.* those of Euclid, excluding his axiom of parallels.

These are: (1) a straight line can be produced indefinitely either way. Take the simplest case, a line through the centre; its "length" is

$$\int \frac{2k^2}{k^2 - r^2} dr, \quad i.e. \quad k \log \frac{k+r}{k-r}.$$

This is infinite when $r = k$, *i.e.* though the line looks finite to us, to the inhabitants it is infinite, just as a meridian is on Mercator's chart.

(2) and (3). One and only one straight line can be drawn between two given points. This is obvious; only one orthogonal arc can be drawn through two given points.

But Euclid's axiom of parallels is not satisfied: draw any orthogonal arc XY (Fig. 1), and take any point, for convenience the centre O. Then clearly

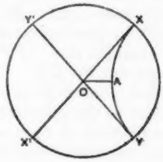

FIG. 1.

through O we can draw as many "straight lines" (or orthogonal arcs) as we please meeting XY, also as many as we please not meeting XY; and the two sets are separated from one another by the two "straight lines" $X'OX$, $Y'OY$, which touch XY at its extremities, *i.e.* at infinity. These we may fitly call the two parallels through O to XY, right handed and left handed respectively. To compare this with Euclid's own form of the axiom, draw OA at right angles to XY; the angle XOA is $< \frac{\pi}{2}$, *i.e.* the two angles XOA, XAO are less than two right angles.

The reader will easily verify for himself the corresponding facts about "planes," *e.g.* that a plane is determined by any three non-collinear points; that two "planes" intersect in a "straight line"; and it will be noticed that all "straight lines" lie in natural planes through the centre. The theory of parallel planes he will find very different from Euclid's.

Congruence. Since our "straight lines" satisfy all Euclid's axioms except that of parallels, figures built up of them ought to comply with all his propositions up to and including I. 28 and all others which do not depend on his theory of parallels, *i.e.* which depend merely on congruence. But how are we to recognise congruence? The theory of reflection in a plane mirror depends on congruence only, hence it ought to be true in our model that the image of an object in a plane mirror is congruent with the object—but for

the difference between right hand and left hand—and that the image is as far behind the mirror as the object is in front. Let us see if this is so.

Take any "straight line" XY (Fig. 2), *i.e.* an orthogonal arc whose centre

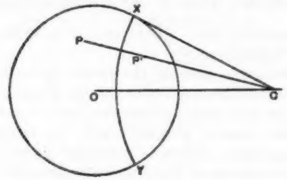

FIG. 2.

is C. Let P be any point, P' its inverse in the arc XY. Let δs be any element of length at P, $\delta s'$ its inverse at P'. Then $\frac{\delta s}{CP}=\frac{\delta s'}{CP'}$. But it is easy to show that $\frac{CP}{k^2-OP^2}=\frac{CP'}{k^2-OP'^2}$;

$$\therefore \frac{\delta s}{k^2-OP^2}=\frac{\delta s'}{k^2-OP'^2}, \quad i.e.\ \delta\sigma=\delta\sigma'.$$

Therefore when any figure is inverted about the line XY (or more generally about the segment or "plane" of which XY is the trace) its inverse is equal to itself, element by element, so far as lengths are concerned; but angles are not altered by inversion; therefore the inverse is congruent with the original (but for right handedness and left handedness). The "plane" therefore behaves as it should in producing an image equal to the object. It is easy to show that the "straight line" PP'—not the Euclidean line, but the orthogonal arc PP'—is bisected by XY.

We have now shown in effect that objects may be moved about without change of shape or size apparent to the inhabitants, and that we can recognise the "equality" in our representation of their world.

The equidistance locus. One particular case is of importance, for it enables us to recognise their "equidistance locus."

Take any "straight line" (Fig. 3)—for simplicity, one through the centre $E'OE$—and two lines $Z'OZ$, $R'AR$ perpendicular to it; let OP be any length on OZ; what will be the equal length on AR?

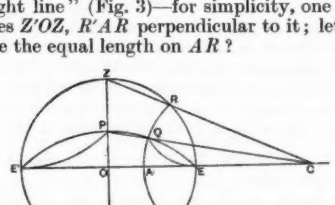

FIG. 3.

Join ZR and produce it and OE to meet in C; C is the centre of a "straight line" which inverts or reflects ZOZ' into RAR', and therefore P into Q; then AQ is "equal to" OP. But $E'PQE$ is cyclic; therefore the locus of Q (P being fixed) is, to us, the circular arc $E'PE$—not of course an orthogonal arc, *i.e.* not a "straight line."

Similarly it can be shown that if XY is any "straight line," the equidistance loci based on it appear in our figure as the other circular arcs through XY;

similarly equidistance surfaces based on a "plane" are non-orthogonal segments of spheres.

But now draw QE the "straight line" parallel to AE, and PE' parallel to OE'; the figure QAE is the inverse of POE', and therefore the angle $AQE=OPE'$.

This shows that the "angle of parallelism" is constant at constant distance. If we take P or Q further away from O or A this angle diminishes, and vanishes when P goes to infinity at Z.

Elementary propositions. Many of the simple but tiresome propositions which are met at the outset of the classical treatment of the subject are now given immediately to intuition; *e.g.* if a line a is parallel to a line b, then b is also parallel to a; if a and b are both parallel in the same sense to c, they are parallel to one another; if a line PQ is a parallel through P to a, it is also a parallel through Q. These need not detain us; but there is one proposition of real importance which must be noted; the classical proof is not very easy, but our model makes it obvious. It is this: if two "straight lines" do not meet, they have a common perpendicular. Translated into our own language it becomes: if two non-intersecting circles cut a third circle at right angles, one and only one circle can be drawn orthogonal to all three.

Rationalisation. The importance of this proposition along with several other points of interest will appear if we consider the problem of describing a "circle" (we do not yet know how their "circle" appears to us) through three points. Euclid's construction and proof of course hold good, provided the two mid-point perpendiculars meet. In his geometry they must meet, as he is careful to prove; but now they may not: what then happens?

If FK, EH (Fig. 4), the mid-point perpendiculars of BA, AC, do not meet, they have a common perpendicular: let it be KH. Draw AX, BY, CZ perpendicular to HK, then it is easy to prove that $AXHE$ is congruent with

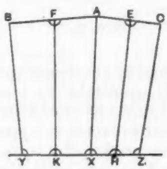

FIG. 4.

$CZHE$, *i.e.* $CZ=AX$. Similarly $AX=BY$; therefore $AX=BY=CZ$, *i.e.* ABC lie on an equidistance locus whose base is HK. This locus takes the place of the circumscribing circle.

Analytically it will be found in such a case that the centre and radius of the circle are imaginary; that the centre is distant $\frac{\pi}{2}ki$ from the line KH and that the radius is $AX+\frac{\pi}{2}ki$.

In Riemann's geometry, a special case of which is ordinary spherical geometry, all these quantities are real: any three points are at once equidistant from a real point and from a real line, and these are distant $\frac{\pi}{2}k$ from one another: a parallel of latitude is at once a circle whose centre is either pole, and an equidistance locus whose base is the equator.

The same phenomena appear whenever we are concerned with the intersections of lines or planes. In Euclid there is only the one case of exception: two coplanar lines either intersect or are parallel, and we get rid of the exception by the use of the analytical term "meet at infinity." In hyperbolic geometry

we have further exceptions: analytically of course we say "meet in imaginary (or 'ideal,' or 'nominal') points," but there is not much satisfaction to the plain man in saying for instance that three points which don't exist are collinear. Useful as this technical sophisticated language is, it is essential also to rationalise it, if the facts are to be understood. That two or more lines meet in an imaginary point means that they have a common perpendicular; that the imaginary intersections of three pairs of lines are collinear (Pascal) will be found to mean that their three common perpendiculars intersect; when the eccentricity and directrix of a conic become imaginary, for the ordinary relation $\sinh SP = e \sinh PM$, we get $\sinh SP = e' \cosh PH$, where e' is real and H is a real *point*.

The angle-sum of a triangle. To return to our model and the main course of development. It is obvious to intuition that the angle-sum of a triangle is $< \pi$, and in a general way that as the triangle grows larger the angle-sum diminishes, and that it may even vanish. The exact relation that the area $= k^2 \times (\pi - \Sigma)$ can be obtained by integration. Take a right-angled triangle in a favourable position, viz. OAP (Fig. 5) and evaluate the integral $\iint \frac{4k^4 r\, d\theta\, dr}{(k^2 - r^2)^2}$ over the curvilinear area OAP, and the result is obtained.

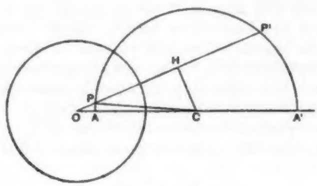

FIG. 5.

This of course would be a highly inconvenient way of measuring areas for all ordinary purposes, and only applicable in practice (as indeed the whole of this geometry would be) to figures of enormous size. For instance, it can easily be shown that if our own universe is hyperbolic, k is not less than say 10^{14} miles; this means that the angles of a square enclosing the earth's orbit differ from right angles by less than one-millionth of a second.

THE TRIGONOMETRICAL FORMULAE.

Most of Euclid's propositions have disappeared with his theory of parallels; there are no rectangles—and therefore no simple theory of areas; no similar figures; the angle in a semicircle is not a right angle; Pythagoras fails us, and the rectangle property of intersecting chords of a circle. We can make no progress till we have found substitutes for these, *i.e.* until we have discovered the new trigonometry.

The development of the trigonometrical formulae is the critical and not the easiest step in the classical treatment. Everyone interested should study it as originally done by Bolyai and Lobatscheffski; but our model yields all that is wanted in the simplest fashion.

We have already obtained the relation between the internal and external measures of a radial line OA; viz. $\rho = k \log \frac{k+r}{k-r}$; which may be written

$$r = k \tanh \frac{\rho}{2k}.$$

In Fig. 5 then, calling OA, OP, r and s or ρ and σ in external and internal measurement, we have $OA=r=k\tanh\frac{\rho}{2k}$, and therefore

$$OA'=\frac{k^2}{r}=k\coth\frac{\rho}{2k};\quad \text{therefore } OC=k/\tanh\frac{\rho}{k},\quad AC=k/\sinh\frac{\rho}{k};$$

and similarly $$OP=k/\tanh\frac{\sigma}{k},\quad PH=k/\sinh\frac{\sigma}{k}.$$

The curvilinear angle $OPA=\angle PCH$; call this β and $AOP=\alpha$, and we have

$$\cos\alpha=\frac{OH}{OC}=\frac{\tanh\frac{\rho}{k}}{\tanh\frac{\sigma}{k}}$$

and $$\sin\beta=\frac{PH}{AC}=\frac{\sinh\frac{\rho}{k}}{\sinh\frac{\sigma}{k}}.$$

Restating these in the ordinary triangle notation and dropping the k for shortness, or taking k as the unit of length, we have: in the triangle ABC right-angled at C,

$$\sin A=\frac{\sinh a}{\sinh c}\quad\text{and}\quad\cos A=\frac{\tanh b}{\tanh c},$$

which easily lead for any triangle to

$$\frac{\sinh a}{\sin A}=\frac{\sinh b}{\sin B}=\frac{\sinh c}{\sin C},$$

and $$\cosh c=\cosh a\cosh b-\sinh a\sinh b\cos C.$$

All the other triangle formulae can be developed algebraically from these; the most important are (for the right-angled triangle)

$$\cosh c=\cosh a\cosh b,$$
$$\tan A=\tanh a/\sinh b,$$
$$\cos B=\sin A\,.\,\cosh b.$$

These are the familiar formulae of spherical trigonometry, circular functions of sides being replaced by hyperbolic.

Having obtained our trigonometrical formulae we can move freely. Instead of Pythagoras we have $\cosh c=\cosh a\cosh b$. From this, for instance, we can prove not merely that the two tangents to a circle from an external point are equal (as we knew already, for it depends only on congruence), but that the tangents to two circles from a point on the common chord produced are equal, or more generally that the radical axis property of any two circles holds good. We can replace III. 35 by the relation

$$\sinh PO\,.\,\sinh OQ\,.\,\cosh^2 CH=\sinh RO\,.\,\sinh OQ\,.\,\cosh^2 CK,$$

where CH, CK are the perpendiculars on the chords; this is useful when we come to the measure of curvature.

By aid of the sine-formula we can replace in a modified form many propositions which we had lost. For instance, VI. 3 becomes

$$\sinh BD:\sinh DC=\sinh BA:\sinh AC;$$

we can re-establish Ceva's theorem with a similar modification, and thereby, as a special case, the intersection of medians. But in the converse of Ceva

we are met again by the complication that the lines may fail to meet—and the proposition needs rationalising accordingly.

We can re-establish the metric treatment of harmonic division—again subject to the necessity of rationalisation when points become imaginary, and so on.

We can investigate the mensuration of solids and surfaces. It is interesting to take some of the important cases—the circle and the sphere—both directly from the model, *e.g.* by integrating $\iiint \frac{8k^6 r \sin\theta\, d\theta\, d\phi\, dr}{(k^2-r^2)^3}$, and by aid of our new trigonometry.

All our formulae can of course be checked by making k infinite and verifying that they reduce to the corresponding Euclidean forms.

Coordinate Geometry. In setting up a system of coordinate geometry it is perhaps as well to avoid using too soon the elegant methods developed by scholars. They will be found to impose themselves naturally as we proceed, and at first, the cruder methods will be found by many more instructive. Taking as coordinates the usual abscissa and ordinate, the equation to a straight line is given at once in the form $\tanh\eta = \tan\alpha \,.\, \sinh(\xi - a)$ or more generally $\tanh\eta = p\sinh\xi + q\cosh\xi$, where for a real line $1+p^2-q^2 \nless 0$.

The necessary expressions for the distance between two points, the angle between two lines, the length of the perpendicular from a point, can be obtained with little trouble. The equation to a circle presents itself first in the form $\cosh\xi\cosh\eta = \cosh a$, which may be written

$$\tanh^2\eta + \sinh^2\xi = \tanh^2 a \,.\, \cosh^2\xi;$$

and similarly the equations to the conics may be developed in quadratic form from $\sinh SP = e\sinh PM$, or from $SP \pm PS' = 2a$.

The geometrical investigation of the conic as the section of a cone is most interesting. The basic property, $SP + S'P =$ constant, is obtained precisely as in Euclidean Geometry; the directrix property, $\sinh SP = e\sinh PM$, will give more trouble—but the trouble will be well repaid. Serious complications of course arise through the non-meeting of lines and planes; but the careful investigation of the various cases (including the rationalisations) is full of interest.

Riemann's Geometry.

If instead of $T \propto k^2 - r^2$ we take $T \propto k^2 + r^2$, we get a model which exhibits the phenomena of Riemann's Geometry. The sphere k is no longer a boundary; we will call it the fundamental sphere. The "straight line" is now a complete circle cutting the F.S. at the ends of a diameter; *i.e.* as in the hyperbolic case, it is a line which inverts into itself about the centre, though now the whole of it "exists," whereas in the hyperbolic case the outer part does not. The "plane" similarly is a sphere cutting the F.S. in a great circle; the F.S. itself is a particular case and an important one, for upon it the scale of measurement is constant, and we can compare our results (so far as they relate to two dimensions) with ordinary spherical geometry.

There are no non-intersecting coplanar lines, and pairs of lines intersect in two points; thus we have abandoned not only Euclid's parallels axiom, but also that which states that there is only one join of two points. A more sophisticated form of Riemann's geometry identifies these two points of intersection, but thereby seems to depart widely from anything that can fairly be called "natural" geometry. If a set of Riemann straight lines and the equidistance loci perpendicular to them is drawn on a central plane, they present the familiar appearance of meridians and lines of latitude as drawn to exhibit the world in hemispheres—but continued outside the bounding circle. They are in fact the stereoscopic projection of the meridians and lines of latitude from a point on the equator, and at the same time are the inversion about this point.

The metric relations developed above for hyperbolic geometry may be

obtained in precisely the same way for this—with of course the substitution of circular functions for hyperbolic; alternatively they may be taken over bodily from the known formulae of spherical trigonometry, but now they apply, with the artificial scale of measurement, throughout space. This space would appear to its inhabitants finite: every "straight line" returns upon itself and is of finite and constant length; similarly every "plane" is of finite and constant area.

In developing the system we miss the complications found in hyperbolic geometry due to non-meeting lines; but instead we have the almost worse complication of lines meeting twice. On the other hand, what is at first sight rather mysterious in hyperbolic geometry often becomes clear and familiar when compared with the corresponding phenomenon in spherical. One example of this has been given already—the fact that a set of points may be at once equidistant from a point and from a straight line. Two others may be given from statics and dynamics respectively.

In hyperbolic statics the resultant of two equal forces P, acting at right angles to and at the extremities of a line $2c$, is $2P \cosh c$, *i.e.* is greater than their sum. But consider the same problem as it occurs on the F.S., *i.e.* on an ordinary spherical surface where measurements are uniform; all the forces and possible motions being restricted to the *surface*, the condition of equilibrium is that of moments about a diameter, and the result is intelligible. (It can also be interpreted in the same way in hyperbolic geometry by use of a pseudospherical surface.) The paradox may be put in an even more effective form thus: the mass of a bar (*i.e.* the coefficient that must be used in the equation) when a bar is struck endwise along its length, is different from that which must be used when it is struck in the middle perpendicular to its length. This again is readily intelligible on the F.S.

The dynamical example is this: consider the motion of a light bar carrying equal masses at the ends, the mid-point being constrained to move along a straight line. If the bar travels without rotation the masses will describe equidistance loci, not straight lines; this requires a force on each normal to its path: there cannot therefore be motion without rotation unless the bar itself is normal to the path, or unless the whole bar lies in the line of motion. There are then two possible positions for irrotational motion, one probably stable, the other unstable, the bar lying either in, or at right angles to, the line along which its centre travels. If the equations of motion are formed, it proves as suspected that the bar oscillates—in the hyperbolic system about the position perpendicular to the line of constraint, in the spherical about that line itself.

To get rid of the oscillation take a system of two bars at right angles carrying four equal masses. We then find that steady motion, *i.e.* uniform travel of the centre and uniform rotation about the centre, is possible—but that there is a constant pressure on the line of constraint. This means that if left entirely free, the system (which we may now replace by a complete ring or a disc) travelling with rotation in its own plane will travel not in a straight line, but in a line of uniform curvature, *i.e.* an equidistance locus, or, if the rotation be rapid enough, in a circle.

Translate this into a disc constrained to move on the F.S. but otherwise free, and we have the familiar problem of a spinning top: the disc if spinning could only travel along the equator under constraint; if free its centre will describe a small circle about a pole, whose position is determined by the relations between the rate of spin and moment of inertia on the one hand and the rate of travel and the radius of the sphere on the other.

Which Geometry is "true"?

The question is still frequently asked: which of these systems, Euclid's, Lobatscheffski's, Riemann's, is true?

To the present writer the answer seems perfectly clear, and he believes that the answer as given below is that which would be given by competent authorities.

They are all true, but though they use the same language, the subject matter of the one is different from the subject matter of the others. The whole question turns on the nature of the straight line.

The straight line being one of the fundamental entities of the subject cannot be directly "defined," for there is nothing simpler in terms of which it can be expressed. We are driven then to indirect definition by axioms. Inasmuch as the axioms of the three geometries differ, it is manifest that they define different things, and it is natural that the developed properties of these things should also differ.

But what, here, do we mean by "things"? Not material objects, with an actual existence in the material world, but ideas, existing only in the mind—suggested by, but not identical with, external objects.

Euclid attempted to put into axiomatic form, that is to say into propositions which others could understand and which could be made the basis of argument, the fundamental properties which he found implied in his idea of "straightness." The failure of the attempt to "prove" the last of his axioms means simply that the definition of *his* straight line (which seems also to be that of the ordinary man) was not complete without this axiom.

In discarding one or more of Euclid's axioms, Lobatscheffski and Riemann in effect proceeded thus: they started with certain axioms, defining they knew not what, though they called it a "straight line," and developed the properties of this unknown entity.

All these geometries are true, but only as applying to the "things" known or unknown to which they refer—just as different algebras may be true.

But when it comes to the application of geometry to the facts of the external world, the question is different. Now we have to ask: which of these absolute pure sciences applies the most conveniently or the most exactly to the facts with which we are dealing? In large scale work, where alone the differences in the results of the geometries are large enough to be apparent, we are dealing chiefly with the form of a ray of light, or the line of action of gravitation. It is easy to conceive that these forms may differ according to circumstances: that while, as seems now to be probable, Euclid's Geometry may be applicable with all necessary exactness to these rays "at an infinite distance" from gravitating matter, Lobatscheffski's, or more probably Riemann's, may afford a better tool for dealing with them in the neighbourhood of such matter.

The essential requisite for clear thinking on the subject is the maintenance of the distinction between the pure science and the applied. The "things" with which the former deals are ideas, abstractions; it can only proceed from axioms, but on that basis its results are absolute. The latter deals with "facts," with "external things"; its basis is experimental and its results approximate.

W. C. FLETCHER.

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161. Definition of Infinity: The place where things happen that don't.—*Per* W. Hope-Jones.

162. Goldsmith hated his tutor Wilder, who was fond of mathematics. Hence the poet's dislike to the subject and his assertion that "all men might understand mathematics if they would."

163. One has merely to read such "splendid atheistic-titanic" deliverances as Mr. Bertrand Russell's "A Free Man's Worship" (cf. *Letters*, ii. 356), to realise that it is precisely because they are so coldly impersonal and remote that mathematics can evoke an emotional response that soothes the fevered spirit.—*Quarterly Review*, No. 468, July 1921, p. 33.

DIFFERENTIALS AS THE BASIS FOR TEACHING THE CALCULUS.

By Alfred Lodge, M.A.

My desire is to emphasize the importance and simplicity of the notion of "differentials" in introducing a student to the calculus. I find that pupils have but little difficulty in realizing the ideas connected with them, and that the notion of a differential coefficient as the ultimate value of the ratio of two connected differentials comes automatically.

A differential in physics and mechanics is a far simpler thing than a differential coefficient, which is an entity different in kind from either of the differentials whose ultimate ratio it represents. For example, in rectangular coordinates, dy, dx are small increments of y and x respectively, but $\frac{dy}{dx}$ is the tangent of an angle.

Again, if y is an area bounded by an ordinate, dy is a small increment of area, but $\frac{dy}{dx}$ is an ordinate.

In mechanics $\frac{dx}{dt}$ is a velocity if dx, dt are increments of length and time respectively, and $d\left(\frac{dx}{dt}\right)$ is an increment of that velocity, whereas

$$\frac{d^2x}{dt^2}\left(\text{short for } \frac{d\left(\frac{dx}{dt}\right)}{dt}\right)$$

is an acceleration.

Thus, if OP, OQ are two successive velocities of a point, the change occurring in time dt, PQ is the change of velocity, and dv, $v\,d\phi$, are components of that change along and perpendicular to the curve respectively, so that $\frac{dv}{dt}$, $v\frac{d\phi}{dt}$ are the tangential and normal accelerations.

Fig. 1.

Fig. 2.

Note that in such a diagram as this, the little triangle PQR is infinitely enlarged, so that OP, OQ are essentially parallel, and therefore OPR is an isosceles triangle with each base angle 90° to the first order of approximation. Similar magnification may be imagined in the dy, dx, ds triangle.

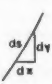

Fig. 3.—Approximate diagram.

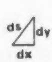

Fig. 4.—Final diagram, infinitely magnified.

Consider, again, partial differentiation, in which the numerator is illuminated by the denominator.

Let $u=c$ be a contour, a function of the east and west coordinates x, and the N. and S. coordinates y.

$\frac{du}{dx}$ is the gradient of the hill in the x direction.

If we want to find $\dfrac{dy}{dx}$, we find $\dfrac{du}{dx}$, where dx is PQ, and du is the height of Q above or below P, and dy is QR, so that in $\dfrac{du}{dy}$ we go from the Q level back to the old level, hence the du of the new fraction is *minus* the previous du;

$$\therefore\ \frac{dy}{dx} = -\frac{du}{dx} \div \frac{du}{dy}.$$

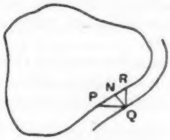

FIG. 5.

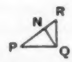

FIG. 6.—Modified diagram.

Again, to find the steepest gradient.
Let QN be perpendicular to the curves, $=dn$ say.

Then
$$\frac{1}{QN^2} = \frac{1}{PQ^2} + \frac{1}{QR^2};$$

$$\therefore\ \left(\frac{du}{dn}\right)^2 = \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2,$$

and the gradient required is

$$\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}.$$

To show that Pv is the rate of doing work where P is the force in the direction of motion, $P\,ds$ is the element of work;

$$\therefore\ P\frac{ds}{dt} \text{ is the rate of doing work.}$$

Or, again, to prove that $P\cos\theta\,\dfrac{ds}{dt}$ (where P is the total force with components X, Y, Z) $= X\dfrac{dx}{dt} + Y\dfrac{dy}{dt} + Z\dfrac{dz}{dt}$, we have the projection of P on the tangent $= X\dfrac{dx}{ds} + Y\dfrac{dy}{ds} + Z\dfrac{dz}{ds}$.

$\therefore$ multiplying this by $\dfrac{ds}{dt}$, the rate of doing work is $X\dfrac{dx}{dt} + Y\dfrac{dy}{dt} + Z\dfrac{dz}{dt}$.

This is clearly the sum of the separate elements of work, $X\,dx$, etc., divided by the time.

Such examples could be multiplied indefinitely, and it is an invaluable aid to students of mechanics and physics to feel that every part of their symbols can be separately considered, and the aggregate made more understandable by such separate consideration.

Illustration from Mechanics—Loney's *Particle Dynamics*, p. 132, Ex. 3.

A particle of mass M moves from rest under a force F constant in magnitude and direction through a stream of fine dust moving in the opposite direction with velocity V, which deposits matter on it at a constant rate ρ. Find the distance moved through by the time its mass is m.

Considering what occurs in the time dt: the opposite momenta of the colliding masses, viz. mv and $-V\rho\,dt$, together with the forward impulse $F\,dt$, give rise to the new momentum;

$$\text{i.e.}\quad mv - V\rho\,dt + F\,dt = (m+dm)(v+dv);$$

$$\therefore\ (F - V\rho)dt = (m+dm)(v+dv) - mv$$

$$= d(mv),$$

where

$$\left.\begin{aligned} m &= M + \rho t; \\ \therefore\ dm &= \rho\,dt; \end{aligned}\right\}$$

$$\therefore\ \frac{F - V\rho}{\rho}dm = d(mv);$$

$$\therefore\ \frac{F - V\rho}{\rho}(m - M) = mv = m\frac{ds}{dt} = m\rho\frac{ds}{dm};$$

$$\therefore\ ds = \frac{F - V\rho}{\rho^2}\left(dm - M\frac{dm}{m}\right);$$

$$\therefore\ s = \frac{F - V\rho}{\rho^2}\left\{m - M - M\log\frac{m}{M}\right\}.$$

Here, the infinitesimals involved all have their definite meanings, and the whole work is alive.

It well illustrates the fact that, in a physical investigation, the initial equations deal with the differentials of the quantities involved, and the relations between the quantities themselves are obtained by integration.

The objection raised to the use of differentials is that they are only approximate increments of a function. Thus, if $y = x^3$, the statement that $dy = 3x^2dx$ is only approximately true unless dx, and consequently dy, be reduced to zero, and therefore it is maintained that in such limiting case the only correct statement is that $\frac{dy}{dx} = 3x^2$. Of course it is true that for actual increments the differential *is* only approximate, but my point is, that it is a good "starter," and in fact a perfect starter for the purposes for which it is generally used, viz. (i) for comparison with another differential, *i.e.* the formation of a differential coefficient; (ii) for integration. The first of these needs no defence, it is obvious: with regard to the second, viz. integration, my contention is that the differential is used as an *indicator*, pointing to the integral function of which it is the differential—a definite function has a definite differential, and a definite differential has similarly a definite function for its integral (barring an arbitrary constant which disappears when limits are inserted). I know that one generally considers the integral as made up of the sum of the differentials, with an error which disappears in the limit, but I contend that we do not really add the imperfect differentials; we merely use them as indicators of the perfect increments which *do* add accurately to the integral function. Thus $\int_a 3x^2dx$ is mentally replaced by $\int_{x=a} \delta x^3$, which is accurately equal to $x^3 - a^3$.

It is true that, as the Greeks seem to have done, we *can* add together the approximate increments, and from the result find the limiting value of their sum, but this is *not* what we actually do when we integrate; we recognize that $3x^2dx$ is the differential of x^3, and therefore x^3 is the function for which we are looking, and $\Sigma\delta x^3$ is what we obtain.

There is hardly need to defend the use of differentials and not differential coefficients in integration. To say that the integral of an ordinate is an area is not a simple idea at all, and is indeed not true, for the other factor, the

differential factor dx, is required; otherwise the result would not be an area but something quite different; an area is obtained by adding up elements of area, not straight lines.

In every case, then, the differential is of the same species as the integral function itself; it is a specimen bit of it, from which we deduce the whole.

So my contention is that a differential coefficient is, in idea, more complex than are the separate differentials whose ultimate ratio it represents, being always different in dimensions from either of them, its meaning being deduced from them, and of course such deduction could not be made if we were not permitted to first think of them separately; the differentials are the stuff out of which the differential coefficient is made, and in integration the differential coefficient does not occur at all.

An early illustration of the simplicity of working with differentials is the formation of $\dfrac{dy}{dx}$ when y is a function of z, which is itself a function of x; say $y=f(z)$, $z=\phi(x)$.

Here $$dy=f'(z)dz \quad \text{and} \quad dz=\phi'(x)dx;$$

$$\therefore \frac{dy}{dx}=f'(z)\phi'(x).$$

It is automatic if we think in differentials.

To conclude with a second-order differential coefficient $\dfrac{d^2y}{dx^2}$, *i.e.* $\dfrac{d\left(\dfrac{dy}{dx}\right)}{dx}$, the two values of dx being taken as equal.

Taking three points P, Q, R on the curve whose ordinates are equidistant, the height of R above the line PQ is the increase of δy, say $\delta^2 y$, and is ultimately d^2y.

$$PL=y,$$
$$QM=y+\delta y,$$
$$RN=y+\delta y+\delta(y+\delta y)$$
$$=y+2\delta y+\delta^2 y;$$
$$\therefore RS=\delta^2 y.$$

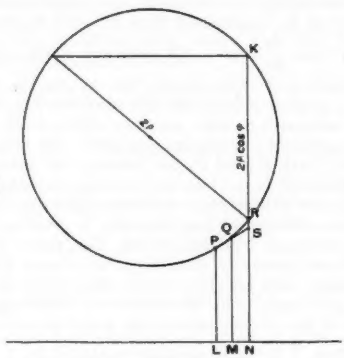

FIG. 6.

[It is a second-order infinitesimal, for $d\left(\dfrac{dy}{dx}\right)$ is first-order; $\therefore \dfrac{d^2y}{(dx)^2}$ is finite.]

The radius of curvature can be easily obtained from the above figure.

For $SR \,.\, SK=SP \,.\, SQ=2PQ^2=2(\sec\phi\, dx)^2$;

$$\therefore d^2y \,.\, 2\rho\cos\phi=2\sec^2\phi(dx)^2;$$

$$\therefore \rho=\frac{\sec^3\phi}{\dfrac{d^2y}{(dx)^2}}.$$

ALFRED LODGE.

A CERTAIN DISSECTION PROBLEM.

BY J. BRILL, M.A.

To cut up an isosceles triangle so that the parts may be put together to form two triangles similar to the original triangle.

Let ABC be the triangle, having the angles at B and C equal, and let D be the middle point of the base. Draw a circle, passing through A, having its centre on AD, and cutting AB and AC in E and F. Join DE, cutting the circle in G.

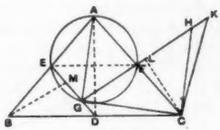

FIG. 1.

Case I. When G lies within the triangle ABC.

Join AG and GF. Then $\angle AGF = \angle AEF = \angle ABC$, and $\angle AGE = \angle AFE = \angle ACB$. Also the angles AFG and AEG are supplementary, and $AE = AF$. Thus, if the triangles AEG and AFG be placed with AE and AF in contact, they will form a triangle similar to ABC.

Produce GF to H. Then CFH = supplement of $\angle AEG = \angle BED$. Also $FC = EB$. Thus, if we make $\angle FCH = \angle EBD$, the triangles FCH and EBD are congruent.

Cut off $HK = GD$, and join GC and CK. Now $CH = BD = DC$, and $\angle CHK$ = supplement of $\angle CHF$ = supplement of $\angle BDE = \angle GDC$. Thus the triangles CDG and CHK are congruent.

Thus the three pieces DBE, CDG and CGF can be put together to form an isosceles triangle CGK.

$$\text{Now } \angle GCK = \angle GCF + \angle FCH + \angle HCK = \angle GCF + \angle EBD + \angle DCG$$
$$= \angle DCF + \angle EBD = \text{supplement of } \angle BAC.$$

To form from this a triangle similar to ABC, it will be necessary to draw a perpendicular CL to GK. Then, cutting along CL and placing GL and KL in contact, we have an isosceles triangle similar to ABC.

If we draw BM perpendicular to ED, then EBM will be congruent with FCL, and MBD with LCH.

Case II. When G lies below the base BC.

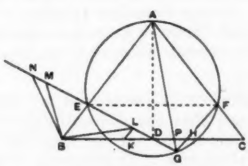

FIG. 2.

Join GA and GF, cutting BC in P and H. The triangles AGE and AGF may be put together to form a triangle similar to ABC, but need the parts DGP and PGH to be supplied. If we cut off $DK = DH$, and draw KL parallel

to GF, to meet DE in L; then DLK is congruent to DGH, and may be cut to provide the missing parts.

If we produce LE to M, and make $\angle EBM = \angle ACB$, and further produce to N, making $MN = KL$, and join BN; we see that HFC will fit on MEB, and BLK on BNM. For $BK = HC = BM$, and $\angle BKL$ = supplement of $\angle LKD$ = supplement of $\angle FHC$ = supplement of $\angle BME = \angle BMN$.

As before, it can be established that $\angle LBN$ = supplement of $\angle BAC$, and this case may be finished off in a manner similar to the first case.

Case III. When G lies beyond the side AB.

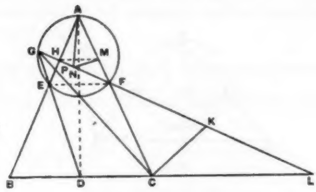

FIG. 3.

Join GC and GF meeting BA in P and H. Produce GH to K and L. Now $\angle CFK = \angle AFG = \angle AEG = \angle BED$. Thus, since $FC = EB$, by making $\angle FCK = \angle EBD$, we have the triangles EBD and FCK congruent. Moreover, $CK = BD = CD$, and $\angle CKL = \angle GDC$. Thus, by making $KL = DG$, we have the triangles CDG and CKL congruent.

Thus the three pieces BDE, GDC, and GCF will make an isosceles triangle with its vertical angle supplementary to BAC. This may be treated as in the former cases. It will, however, be necessary to obtain the missing parts GEP and GPH from AHF.

Make $FM = EH$ and $FN = EG$, and join AN and NM. Then, since $\angle AFG = \angle AEG$, the triangle FMN is congruent to GEH, and will supply the missing pieces.

Further, since $\angle NMF = \angle GHE$, the quadrilateral $AHNM$ is cyclic; and since $AH = AM$, and HM is parallel to BC, it is easily shown that the triangles AHN and AMN can be put together so as to make a triangle similar to ABC.

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164. I doubt very much whether there is any advantage to be derived from the introduction of algebra into the grammar school course. To the ordinary citizen in practical life a knowledge of algebra will be found of little value. No one buys or sells by algebra; and a knowledge of polynomials or the quadratic equations would be of little use to the housewife in the discharge of her duties. . . . I urge therefore in place of algebra that the beautiful system of arithmetical thought known as mental arithmetic be more fully introduced into your schools than it is to-day.—Dr. Brooks, of Philadelphia (*circ.* 1892).

165. Admiral Peary being cross-questioned by the Hon. Mr. Macon of Arkansas: " For the reason that the diminishing centrifugal action, and in proportion, the increasing centre of gravity near the pole causes a complete failure of man and animal energy that produces a kind of paralysis of the senses and of motion, a paralysis of sensation in any part of the body, including the exercise of the faculty of the mind. . . . So that it would be almost impossible for them to exercise their independent functions so that anybody could ascertain a real fact, intelligently ascertain a fact." To which Peary replied, " I have never heard of that before."

MATHEMATICAL NOTES.

655. [V.] *Query: the earliest appearance of* $\therefore$ *for "therefore" and of* $\because$ *for "because."* M.

[To save time it is useful to mention upper and lower limits. Dodson's *Mathematical Repository*, 1748, uses Th. for "therefore." Donn, in *A New Introduction to the Mathematics*, 1758 (p. 3), gives $\because$ as a "character" representing "therefore," and the Gentleman's *Mathematical Companion*, 1805, uses $\because$ for "because." The *Diaries* seem to use $\therefore$ for "therefore" from 1800 sparingly, and more generally in and after 1806. As late as 1846 D. Lardner's *Euclid* says of $\because$, "this sign expresses the word 'therefore.'"]

The above query may be extended to the following. Though Harriot used $>$ and $<$ about 1610, Donn (*loc. cit.* p. 7) uses ⊏ for $<$ and ⊐ for $>$; and as late as 1742 James Bernoulli (or his printer) uses the capital V, $\prec$ and $\succ$. The capital letter saved cutting a special type. Thus Donn uses the capital S, in $a \backsim b$, where we now write $a \sim b$; and the *Companion* (*loc. cit.*) has

$$\sin\frac{AC \backsim BC}{2}.$$

James Bernoulli, 1742, and Euler, 1748, use lx for $\log x$. Smith (of Smith's Prize fame) explains as follows a form used by Cotes in his *Harmonica Mensurarum.*

Si R sit radix quadratica quantitatis affirmativae, adeoque possibilis: erit $R \left| \frac{R+T}{S} \right.$ mensura rationis quam habet $R+T$ ad S pro Modulo R. Sin R sit radix quadratica quantitatis negativae, adeoque impossibilis: assumi debet pro R radix quadratica ejusdem quantitatis affirmativae sumptae, eritque $R \left| \frac{R+T}{S} \right.$ mensura Anguli ad Modulum R, cujus Radius, Tangens & Secans sunt inter se ut R, T, & S.—Smith, *Notae ad Harmoniam Mensurarum*, 1722 (p. 97), *Opera Miscellanea*, Rogeri Cotes.

656. [V. 1. a. ζ.] *"Complete Angle" or "Cross"?*

The usefulness in elementary geometry of the conception of an angle which is familiar in advanced work is beyond dispute, and Mr. Picken's article on p. 188 of this volume deserves the attention of every teacher of the subject. But perhaps the name "complete angle" is unhappy. There is no analogy with "complete quadrilateral," for in the latter phrase the reference is not to the fact that the sides are not terminated but to the fact that the six vertices are all recognised. The name of "ordered line-pair" explains itself, but is there any reason why we should not speak simply of the "cross" (l_1, l_2)?

There is less difficulty than Mr. Picken seems to suppose in proving the angle property of the circle. If A, B, C, D are concyclic, and if AD, BC cut in O, then $OA \cdot OD = OB \cdot OC$, that is, $OA : OB = OC : OD$; hence the triangles OAB, OCD are *inversely* similar, and the crosses OBA, ODC are *inversely* congruent. That is, the crosses ABC, ADC are *directly* congruent. This proof fails if AD, BC are parallel, but of the three crosses through four points, at least one must have an accessible centre.

It is true that the proof reverses the order of dependence between the angle property and the rectangle property which is now usual, but (1) direct proof of the rectangle property is easy, (2) the fact that in an isotropic plane the rectangle property remains significant but the angle property collapses, seems to indicate that the former occupies the simpler position logically, and this view is confirmed by examination of the extensions to conic sections.

E. H. Neville.

REVIEWS.

The Quantum Theory. By F. Reiche. Translated by H. S. Hatfield and H. L. Brose. Pp. 183. 6s. net. 1922. (Methuen.)

The central idea of this revolutionary theory is that in certain circumstances changes of energy are not continuous. When such a change takes place its amount cannot be less than $h\nu$, where ν is the frequency of the waves of light or heat corresponding to this change, and h is a constant (Planck's constant). More generally, the amount can be any integral multiple of this minimum, which is called the *quantum*. This hypothesis had its origin in Planck's efforts to construct a theory which would agree with observed facts concerning the radiation of heat, but it was afterwards applied to many other branches of physics, in particular to the theories of specific heats at low temperatures, of photo-electricity (the emission of electrons from a body under the action of light), and of the spectral lines characteristic of the chemical elements. There is much that is obscure and unsatisfactory in these theories, but the striking fact is that the experimental results seem to demand an energy quantum in each case, and, what is more, demand approximately the same numerical value of h.

Perhaps the best way of introducing the quantum theory to those who have not met it before is to discuss Bohr's theory of spectral lines in the simplest case possible. Well-attested experiments have shown that if an electric discharge takes place through hydrogen, sharp lines are observed when the gas is viewed through a spectroscope, and the frequencies of the light corresponding to these lines (deduced from the distances between them) are given by the remarkable formula

$$\nu = R\left(\frac{1}{n_1^2} - \frac{1}{n^2}\right),$$

where R is a constant (Rydberg's constant) and n and n_1 are *positive integers*. For instance, putting $n_1=2$ and $n=3, 4, 5, 6, \ldots$ we get what is known as the *Balmer Series*. Following Rutherford, Bohr supposed the hydrogen atom to consist of an electron (a particle of mass m with a negative electric charge $-e$) revolving round a nucleus (another particle of very much greater mass with a positive charge e). The common centre of mass is so near to the nucleus that to begin with we shall take the nucleus as fixed. When the electron is describing a circle of radius a with angular velocity ω under the electric attraction e^2/a^2 between the electron and the nucleus, we get, by equating this attraction to the product of the mass and the central acceleration, the equation

$$ma\omega^2 = e^2/a^2.$$

This equation by itself certainly does not lead us to the result required. It is natural to assume that ν is the frequency of the circular motion, so that $\nu = \omega/2\pi$, but nothing is gained thereby, and the assumption seems to be quite unfounded.

To obtain other equations, Bohr made the following assumptions :

(*a*) Atoms exist normally in one or another of certain states called *stationary states*. While the atom is in one of these energy is neither emitted nor absorbed.

(*b*) In the transition from one stationary state to another, energy is emitted or absorbed according to the quantum law

$$W - W_1 = h\nu \text{ (emission)}, \quad W_1 - W = h\nu \text{ (absorption)},$$

where W is the total energy (kinetic and potential) of the atom in its first state and W_1 that in the second.

(*c*) The angular momentum $ma^2\omega$ has one of the values $nh/2\pi$, where n is a positive integer. From the simultaneous equations

$$ma\omega^2 = e^2/a^2, \qquad ma^2\omega = nh/2\pi,$$

we obtain
$$a = n^2h^2/4\pi^2me^2, \quad \omega = 8\pi^3me^4/n^3h^3.$$

Hence
$$W = \tfrac{1}{2}ma^2\omega^2 - e^2/a = 2\pi^2me^4/n^2h^2 - 4\pi^2me^4/n^2h^2 = -2\pi^2me^4/n^2h^2.$$

Finally from $$W - W_1 = h\nu,$$

we get $$\nu = \frac{2\pi^2 me^4}{h^3}\left(\frac{1}{n_1^2} - \frac{1}{n^2}\right),$$

which is of the form required. What is more, the constants R and $2\pi^2 me^4/h^3$ agree very closely in numerical value.*

Similar calculations apply to other elements. A slight discrepancy in the values for R is explained when we take into consideration the motion of both nucleus and electron about their common centre of mass. Elliptic orbits were dealt with by Wilson and Sommerfeld, who replaced Bohr's (or Nicholson's) angular momentum assumption by the more general form

$$\int p_s dq_s = n_s h, \quad (s = 1, 2, 3, \ldots f),$$

where q_s is a suitable *generalised coordinate* (as used, for example, in Lagrange's equations),

p_s is the corresponding *generalised momentum* $\partial T/\partial q_s$ (T being kinetic energy),

n_s is a positive integer, f is the number of degrees of freedom of the system,

and the integration is extended over the complete period of q_s.

This generalised theory has been brilliantly successful in explaining many otherwise obscure phenomena, such as the *Stark effect* (the splitting of a single spectral line into several by the action of an electric field), and, in combination with the theory of relativity, the *fine structure* of the hydrogen lines. However, it must not be overlooked that the assumptions of Bohr's theory are in flat contradiction to ordinary mechanics and physics. It would almost seem as if we at times use these ordinary ideas and at other times reject them, whichever is most convenient at the moment !

Planck's original treatment of the problem of heat radiation from "black" bodies was open to the same objection. He used a combination of thermodynamics and electromagnetic theory to obtain a result inconsistent with that theory. This result (Planck's law) is

$$K_\nu = \frac{h\nu^3}{c^2} \cdot \frac{1}{\left(e^{\frac{h\nu}{kT}} - 1\right)},$$

where T is the temperature, c the velocity of light in a vacuum, k a constant, and K_ν a quantity which needs very careful definition, but which we may say roughly is a measure of the emissive power of radiation of frequency ν. In spite of the weaknesses of Planck's theory, there can be no doubt that his law agrees closely with experimental facts, while the Rayleigh-Jeans law (which can be derived from Planck's law by taking the limit as h approaches zero, that is by taking energy changes as continuous) is quite opposed to these facts. Moreover, Poincaré has shown that the quantum hypothesis is the only one that leads to Planck's law. Originally Planck believed that his law held both for emission and absorption. Later on, to meet some serious objections raised by Lorentz, Planck took absorption to be continuous and only emission to be governed by the quantum law.

Einstein (whose activities are by no means limited to relativity) applied Planck's law to explain the diminution of specific heats as the temperature decreases. His theory did not meet the whole of the observed facts, and it is now replaced by more elaborate ones (on similar lines) due to Debye and others. Einstein's name is attached to the principal law of photo-electricity

$$\tfrac{1}{2}mv^2 = h(\nu - \nu_0),$$

* Many books use the word *frequency* in its correct sense for theoretical equations, but also use it (incorrectly) as meaning *the reciprocal of the wave-length* for experimental results. If this is done, our expression for R must be divided by c, the velocity of light in a vacuum, since c=wave-length × frequency.

where v is the maximum velocity of the electrons, each of mass m, emitted from the surface of a body when light of frequency ν falls on it. Unless ν exceeds a certain minimum value ν_0 no emission takes place. Einstein's *light-quanta hypothesis* is that the quantum condition holds good for light even when it is travelling in free space. This hypothesis has some great advantages, but it is not accepted by Planck, and it is difficult to reconcile it with the observed facts (such as interference phenomena) with which the ordinary wave theory of light agrees so well. Here, as elsewhere, the quantum theory is full of difficulties and apparent contradictions. There is certainly ample scope for further research.

The subjects mentioned hitherto form only a portion of those dealt with in Professor Reiche's interesting book, which covers a wide range. He deals in some detail with X-rays, the kinetic theory of gases, and the molecular theory of solids. Perhaps the chapter dealing with solids is unduly long and out of proportion with the very brief treatment of photo-electricity. This is regrettable, for the most trustworthy determination of h is that made by Millikan from photo-electric measurements. A valuable feature of the book is the very full list of references to original researches, though the pioneer work of Nicholson and Wilson is not mentioned. There are also several mathematical notes, but these are not altogether satisfactory. They are unnecessary for those who require only the results without the arguments on which they are based, and yet they are too disconnected for those who wish to follow the reasoning. The careful reader will have terrible difficulty in places, most of all in the first chapter. The definition on p. 2 is incorrect, and the notation (perhaps due to a misprint) is such as to clash with later pages. Parts of the derivation of Planck's law are given very fully, but other essential parts are not given at all. Of course Planck's own argument (given in his *Heat Radiation*) is very long and involved, but the much shorter proof due to Debye (pp. 12-15 of Wien's *Neuere Probleme der theoretischen Physik*) could have been given in full. However, the book is a very useful one, and it contains much that is not to be found elsewhere except in a foreign tongue. Several experimental physicists have found it very instructive, and the mathematician will find it a good summary of results which can be followed up in the references given. The publishers deserve our gratitude for their enterprise in presenting an English version, and it is to be hoped that they will not keep us waiting much longer for their translation of Sommerfeld's *Atombau und Spektrallinien*, the standard treatise on spectral lines.

H. T. H. PIAGGIO.

Exponentials Made Easy or The Story of 'Epsilon.' By M. E. J. GHEURY DE BRAY. Pp. x+253. 4s. 6d. net. 1921. (Macmillan.)

This fascinating little brother to *Calculus Made Easy* is dedicated to the memory of Dr. S. P. Thompson, and emulates the captivating style of its predecessor. It is a very stimulating book, and brings out the various properties of ϵ with a forcefulness which cannot be denied. Part I. deals with functions, indices, logarithms, the radian, and infinite series (chiefly binomial theorem). Part II. introduces ϵ as $\operatorname{Lt}_{n\to\infty}(1+1/n)^n$, which is identified with the growth-factor in *continuous* compound interest for the time in which a sum of money would double itself at simple interest. The exponential and logarithmic series are then deduced from the binomial theorem, and ϵ is discerned at home in the logarithmic spiral, in the hyperbola, in the "slack rope," and in the path of the focus of a parabola which rolls along a fixed tangent. The last two chapters are devoted to the probability curve and exponential analysis.

Of course one does not look for mathematical rigour in this treatment, but there should be no serious mathematical errors. Some are merely slips or misprints which can easily be corrected in future editions. But others cannot be so readily excused. For instance—on page 143, when $n\to\infty$ it is assumed that $\sqrt[n]{x}\to 0$, x being positive and >1; on page 147, it is assumed that the centre of gravity of a uniform curved chain is at the middle point of the chain; on page 158, it is stated that "ϵ is not a mere number, but a definite length."

Probably the binomial theorem still occupies too prominent a position in the popular presentation of infinite series, and teachers would be well advised

to consider alternative methods of approach. The earlier introduction in recent years of common logarithms and antilogarithms might perhaps be utilised to give a more natural introduction to ϵ. Wherever an arithmetic series of values in one variable is accompanied by a geometric series of values in a dependent variable ϵ is to be found. In the exponential curve $y=C.a^x$, ϵ is the ratio of two ordinates whose distance apart is $\log_a\epsilon$, which distance is also the constant sub-tangent. In the spiral $r=C.a^\theta$, ϵ is the ratio of two radii inclined at an angle $\log_a\epsilon$, which is also the tangent of the constant angle at which the curve crosses the radius. In the hyperbola $y=a^2/x$, ϵ is the ratio of two abscissae between which the ordinate generates an area a^2. As an introduction to the hyperbolic functions, the properties of the two functions $\frac{1}{2}(a^x+a^{-x})$ and $\frac{1}{2}(a^x-a^{-x})$ may well be discussed, illustrating from $a=2$ and $a=10$ and bringing out the advantage of substituting ϵ for a.

Plane Geometry for Schools, Part I. By T. A. Beckett and F. E. Robinson. Pp. viii+239. 5s. 1921. (Rivingtons.)

This first part comprises a preliminary section followed by Sections I., II. and III. In the preliminary section the standard geometrical constructions are set out without proof. Then, following a few pages on simple areas and simple solids, formal definitions are given to form part of the basis of the deductive geometry in the subsequent sections. The difficulty of dealing with parallels is got over by defining parallel straight lines as "those which are in the same plane, and make equal angles *in the same sense* with a straight line cutting them"; and it is stated without proof that such "lines are in the *same direction*, and will not meet at any finite distance from the transversal."

Section I. deals with axioms (including Playfair's) and postulates, angles, parallels, congruent triangles, parallelograms, projections and loci, the definition of parallels, quoted above, forming the basis of a short and easy sequence.

Section II. deals with the circle; Section III. with areas, Pythagoras' Theorem, Coordinates and contour lines, gradient and introductory trigonometry.

There is throughout an abundance of well-chosen and carefully graduated exercises.

The book is intended "to provide courses of deductive and practical geometry, combined in such a way as to emphasise their interdependence, without sacrificing the logic of the one or neglecting the importance of the other"; and the authors claim to have "devoted special care and attention to" the subject of Loci. On page 99 it is very properly pointed out that "In order to identify a line as a locus, it is necessary to show—(i) that any and every point on the line must satisfy the given conditions; (ii) That a point which satisfies the given conditions must be on the line." Now this is exactly what the authors fail to do, at any rate in the text. For instance, on page 100 it is proved that if $PA=PB$, and if M is the middle of AB, A and B being two given points, then PM is at right angles to AB. This conclusion is interpreted as proving that the locus of P is the perpendicular bisector of AB. Thereafter this proposition is freely used. On the next page it is quoted as proving that if YO bisects AC at right angles at Y, then $OC=OA$. On page 117 it is quoted as proving that a perpendicular from the vertex of an isosceles triangle upon the base bisects the base.

In Section III. areas are treated as measurable, the expression

$$\tfrac{1}{2}(\text{altitude})(\text{base})$$

for the area of a triangle being used to prove the proposition that "triangles on equal bases and between the same parallels are equal in area," from which the corresponding proposition concerning parallelograms is deduced.

The book contains much good material, but will need considerable revision before it can profitably displace many existing text-books.

Plane Geometry, Practical and Theoretical Pari Passu. By V. Le Neve Foster. Vol. I., Pp. xi+228+x; Vol. II., Pp. xi+196+viii. 3s. each volume. 1921. (Bell.)

This work, manifestly that of a teacher, exhibits several outstanding features;—it contains a wealth of illustration in applications to everyday

affairs, is written and printed with unusual care, and is profusely illustrated with excellent diagrams. The subject-matter is made to stand out clearly by a free use of clarendon type; and throughout, bookwork and examples are indicated as appropriate to first, second or third readings. The numerous exercises are separated into examples of geometrical drawing, or calculations, or riders. Every possible pitfall is anticipated, and every possible help given to the learner, so that there is very little left for a teacher to do except to see that the pupil uses the book.

In the early stages a foundation is laid by means of careful geometrical drawing, appeal being made to common-sense and intuition. The first formal proposition is Euclid I. 32, and appears on page 53. The work does not claim to provide a system of deductive geometry founded on a few axioms, and frequently appeals to intuition, but it contains good examples of deductive reasoning carefully set out. Volume I., consisting mainly of exercises, contains 20 formal propositions, and its scope comprises straight lines, angles, triangles, inequalities, polygons, areas (with Pythagoras' theorem) and loci, the last two items in particular being extremely well treated; but the author seems afraid to use in Volume II. the loci established in Volume I. Volume II. is somewhat more concise, and contains twenty-six formal propositions. Its scope comprises circles, rectangles and similarity. The proofs of propositions 22 and 25 are incomplete.

Elementary Statics of Two and Three Dimensions. By R. J. A. Barnard. Pp. 254. 7s. 6d. 1921. (Macmillan.)

An excellent treatise for senior students,—concise, clear, well arranged, and illustrated with numerous examples. The topics friction, work and machines are introduced early, and naturally; and there is a short, but illuminating, chapter on shearing stress and bending moment. The book also contains brief, but adequate, introductions both to graphical statics and to analytical statics of two and three dimensions, and the infinitesimal calculus is employed in the chapters on virtual work and centres of mass. The last chapter treats of vectors in space, introducing Poinsot's central axis. It is pleasant to find so much information in so small a compass.

There are several errors, both in the text and in the diagrams, some of which are likely to prove rather disconcerting to a beginner. These can easily be corrected in a future edition, but in the meantime it would be well to insert a list of *errata*.

A more elegant geometrical construction for the centroid of a quadrilateral area $ABCD$ (whose diagonals meet at O) might with advantage be appended to §106:—Along DB take $DE = OB$; then the centroid required coincides with that of ACE.

W. J. Dobbs.

Life Contingencies. By E. F. Spurgeon. Pp. 477+xxviii. 30s. 1922. (Published by the authority and on behalf of the Institute of Actuaries by C. and E. Layton.)

Of recent years many books have been produced in this country, in America and on the Continent, dealing with "probability," and with the mathematical developments that are connected with statistical work. Much of this work consists of descriptions and criticisms of what might be described as "theories" of probability, and it is interesting to take up a book such as Mr. Spurgeon's on Life Contingencies, which ignores these refinements and merely takes the simple rules of probability and evolves from them what may fairly be described as a complete treatise on the elements of life contingencies and the calculation of monetary values for those complicated benefits depending on them with which actuaries have to deal. Mr. Spurgeon assumes in fact that the actuary knows with sufficient accuracy the probability that a person of any given age will survive one year. Whatever theory we may hold with regard to the probability that an event will happen on a further trial when it has already happened in n cases out of m trials, we must admit that no other initial assumption than that of Mr. Spurgeon's is likely to be satisfactory in actuarial practice.

The main object of the book is to provide a text-book from which actuarial students can obtain a satisfactory knowledge of what is perhaps the most

important part of their preliminary studies. It would, however, have been unsatisfactory if such a book had been one of those treatises which merely enables a well crammed student to pass difficult examinations, because in the present instance the student will not only have to acquire the knowledge for the time being, but will want to keep it throughout his professional career.

After dealing briefly with the construction of tables of the rates of mortality, Mr. Spurgeon discusses the methods of calculation that have been adopted for obtaining benefits depending on a single life. Having in this way given an idea of the subject in its simplest form, he discusses annuities payable more frequently than once a year, and annuities payable with a final proportion to the date of death. These chapters appear to be so simple that one is surprised by the remembrance that they relate to subjects that students have always found difficult. Mr. Spurgeon then turns to life office valuations, which really lie outside the scope of his work, but are inserted in outline in order to show the student where a part of the subject is leading him, and then we find two chapters dealing with the Gompertz-Makeham law of mortality, and the statistical applications of the mortality table. The former of these two chapters verges a little on the theoretical, but so many useful approximations to complicated functions result from the assumption of these "laws" that it is important to give them, even although it is unsuitable at such an early stage of the actuarial student's career to bring him into touch with the graduation of mortality tables. The chapter on statistical application is excellent, and we wish that, with a few additional warnings, parts of it could be published as a tract and put in the hands of those people who have had no actuarial training, but who write about "expectation of life" and "death rates" without appreciating the limitations of either. This concludes part I. of Mr. Spurgeon's book, and in part II. he deals with functions involving two or more lives, starting with probabilities, and then going on to annuities and assurances where more than one life is involved, and finally to contingent assurances and reversionary annuities. Throughout this part of the work the method followed by Mr. Spurgeon is to show the true value for each annuity or assurance on the assumption that the probabilities of life at each age are known, and then to show in each case how approximations to these exact values can be found in practice. In nearly every case the exact expression for the value of an assurance is given in the form of an integral, and the student is shown how to reach arithmetical results by ordinary methods of approximate integration. In several numerical examples Simpson's first rule is used, but in others Mr. Spurgeon has adopted a formula which, in the form in which he uses it, will, we think, be rather a shock to a mathematical reader. The formula in question is expressed as follows:

$$\int_0^\infty f(x)\,dx = n\{\cdot 28f(0) + 1\cdot 62f(n) + 2\cdot 2f(3n) + 1\cdot 62f(5n) + \cdot 56f(6n) + 1\cdot 62f(7n)\},$$

and is found by taking

$$\int_0^{6n} f(x)\,dx = n\{\cdot 28(f(0)+f(6n)) + 1\cdot 62(f(n)+f(5n)) + 2.2f(3n)\}$$

and adding two terms from a second integral. It has become popular in actuarial circles for no sufficient reason. If the interval, n, be properly chosen, it would be equally legitimate to include a different number of terms in the second integral: in fact Mr. Spurgeon actually does this in some examples when $f(7n)$ happens to be negligible. If the author felt that he could not break away from tradition, he should have shown clearly the point indicated: possibly he would say that actuarial students have already appreciated such points before they read life contingencies; this is true, but it does not deal with the difficulties of the reader who has not been brought up in the particular school and finds this strange formula confronting him unproved and unexplained.

The book concludes with a discussion on the construction of tables, and particularly monetary tables, made from the probabilities of life ascertained from statistical experience and with a chapter showing how tables are made in

practice when two or more causes of decrement are involved, such as mortality and withdrawal or retirement. The last eighty pages of the book are devoted to tables based on the English Life Table No. 8, H^M table (Makeham Graduation), and the $O^{[NM]}$ table. The last is given in the form of a "select table," that is a table showing probabilities and monetary values in terms not only of the attained age, but also of the time elapsed since a life was selected for assurance. These tables are well chosen. Our only criticism of them is that in certain cases approximate values are given for the populations living between ages x and $x+1$, and for complete expectations of life, although it would have been comparatively easy to give values exact to the number of figures published.

Mr. Spurgeon obviously appreciates the student's difficulties; he shows what is the use of the thing that is being learnt, where the difficulties lie and how to overcome them, and he has succeeded in producing a book which, although necessarily a compilation, gives an impression of freshness and almost of originality.

There are a few misprints and mistakes, but they are unimportant. It is, in fact, a good book, and the author, printer (Cambridge University Press), and the Institute of Actuaries, on whose authority it is published, are to be congratulated. It should appeal to many readers outside the particular circle for which it is primarily intended. W. PALIN ELDERTON.

Analytic Geometry. By LEWIS PARKER SICELOFF, GEORGE WENTWORTH and DAVID EUGENE SMITH. Pp. 288, with Index. 11s. 6d. net. 1922. (Ginn.)

This is one of a well-known series of elementary mathematical works published in America by teachers whose avowed purpose is to produce in a handy form books which embody the best thought on mathematical teaching. First let the scope of the book be explained.

The book is intended for a year's work at college (presumably this includes schools), for beginners. "It is not so elaborate in its details as to be unfitted for practical classroom use; neither has it been prepared for the purpose of exploiting any special theory of presentation; it aims solely to set forth the leading facts of the subject clearly, succinctly." We agree that this is borne out by the book itself, and with a few slight reservations can heartily recommend it for general use.

The subject matter includes an account of the straight line, circle and conic sections, leading on to a thoroughly readable introduction to the classical curves of higher orders, and quadric surfaces. Historical references to the solution of problems, which are insoluble with ruler and compass, are neatly made. Curve tracing is considered from the outset, and general principles are unfolded before the reader is taxed with great detail of formulae.

The exercises are good, especially the practical exercises, where the methods of this geometry actually have practical use; also such as this: "How does the graph of the equation $\rho=a\cos\theta$ differ from that of $\rho^2=a\rho\cos\theta$?"

The logical treatment is generally good, and suggests that to be up to date is not necessarily to be slipshod or garrulous. The figures are numerous and well drawn; the use of polar coordinate mesh printing is a happy alternative to the ordinary eye-straining squared paper.

The few faults which can be discovered might easily be set right. There is no definition of a *limit*, although this term is used on p. 48; and the old error of confusing the limit of a function when $x\to a$ with the value of the function when $x=a$ has still survived in § 49. The use of positive and negative coordinates is not well expressed, for §§ 16 and 220 contradict each other. If O is origin and A has coordinates $(-1, 0)$, § 16 implies that the segment OA is negative; § 220 implies that it is positive.

A little more might be said on p. 210 to explain the connexion between Cartesian and polar coordinates.

The use of parameters is deferred to p. 224: these might take a more important place in the work.

There are no forward references to urge the student to read in advance. The book is good enough to act as a stepping stone to larger treatises.

(1) **Mann and Norman's Algebra.** By H. T. Mann and J. S. Norman. Pp. 262+vii. 4s. net; with Answers, 4s. 6d. net. 1922. (H. F. W. Deane & Sons, The Year Book Press, London.)

(2) **Elementary Algebra. Part I.** By E. H. Chapman, M.A., D.Sc. Pp. 143+xxii (Answers). 3s. net. 1922. (Blackie & Son.)

These two books have been written for beginners. They cover the ground up to and including quadratic equations. So many books of like scope have appeared recently that they inevitably challenge comparison. Neither of these books makes notable departures from common practice. They embody detailed improvements which have lately come to be widely accepted.

It would be interesting to see a bolder use of equations as a starting point. To the reviewer the equations

$$2x+4=8 \quad \text{and} \quad 2x+4=x+6$$

are of quite distinct educational value. The first equation is no fair example of algebra; it is disguised arithmetic. The second, however, contains the germ of algebra. It contains something quite in advance of ordinary arithmetic, and at once appeals to the intelligent beginner.

There are occasional lapses into loose wording, *e.g.* on p. 221 of book (1), where maximum and minimum are introduced to qualify the value of a parabola, not of its ordinate. The writers have not attempted to press the importance of a *variable*.

Messrs. Deane & Sons have printed their book extremely well.

H. W. Turnbull.

Haupt-Exponents, Residue Indices, Primitive Roots and Standard Congruences. Lt.-Col. Allan Cunningham, R.E., Mr. H. J. Woodall, A.R.C.Sc. and (the late) Mr. T. G. Creak. Pp. 136. 1922. (Francis Hodgson, London.)

The haupt-exponent of the base y (mod p), p being a prime number, is the least exponent ξ such that

$$y^{\xi}-1\equiv 0 \pmod p.$$

From Fermat's theorem it follows at once that ξ is a factor of $p-1$. The first table in the book before us gives the haupt-exponents of the eight bases, 2, 3, 5, 6, 7, 10, 11, 12 for all primes under 10000. It is convenient to tabulate $\nu\equiv(p-1)/\xi$ also, and in most cases ν is an integer of one digit. The values of ν only for the same eight bases are tabulated as far as $p=25409$.

Within the limits of the tables prime factors of $y^{\xi}-1$ can be obtained by inspection. It is only necessary to run the eye through the y columns in search of the appropriate ξ, and a prime factor appears on the left.

When $\xi=p-1$, as often happens, y is called a primitive root of p. In this case the remainders, when

$$y,\ y^2,\ \ldots\ y^{p-1}$$

are divided by p, are all different. The table contains a note of the least positive and least negative primitive root of each of the 2800 primes to 25409. In 2637 cases the least positive primitive root does not exceed 12, and is taken at once from the table of haupt-exponents, and, among the rest, the largest obtained is 31. Of the least negative primitive roots $-g$, 2688 values of g do not exceed 12 and the greatest of the rest is 39. Since the existence of primitive roots was first noticed by Gauss various tables of them have been compiled, but the present one is far in advance of anything done before it.

The volume also contains tables giving the least powers of 3, 5, 7 and 11 congruent to powers of 2 and 10 (mod p) for each prime p less than 10000. Typical entries are

$$\left.\begin{array}{llll} 2^{34}\equiv-3^{14}, & 2^{17}.3^{14}\equiv1, & 2^{46}\equiv-5^{4}, & 2^{5}.5^{4}\equiv1, \\ 10^{130}\equiv7, & 10^{74}.7\equiv-1, & 10^{69}\equiv11^{7}, & 10^{115}.11^{7}\equiv-1 \end{array}\right\} \pmod{2857}.$$

Combinations of these standard congruences are of frequent use in finding haupt-exponents and primitive roots.

The results of about 30000 arithmetical calculations, many of them laborious, are contained in the book. Of course, in computations of this type consecutive entries cannot be checked by differences as in a table of sines or logarithms. Much credit is due to Lt.-Col. Cunningham and his collaborators for the care they have bestowed on the calculation of these tables. Each entry has been worked independently by two or more hands. W. E. H. B.

Frequency Arrays, Illustrating the Use of Logical Symbols in the Study of Statistical and other Distributions. By H. E. Soper. Pp. 48. 3s. 6d. net. 1922. (Cambridge University Press.)

It seems to the present writer that history does not favour the author's expectation that the ideas he puts forward will be of great service in statistical developments. But all serious students beginning to read the mathematical theory of statistics should study this interesting little book. It will help them to gain a broad detached outlook essential to the understanding of fundamental possibilities.

A considerable knowledge of pure mathematics is assumed at times, and in a rather off-hand manner. As the author does not profess to be concerned with merely mathematical processes, no doubt he is justified. Still, he might have been just a little more considerate of the feelings of the mathematicians!

J. F. Clatworthy.

Conférences sur les Transformations en Géométrie Plane. Par W. de Tannenberg. Pp. 50. 4 fr. 1922. (Vuibert.)

The transformations considered are those of Translation, Symétrie (Reflexion in a straight line), Rotation, Homothétie (Stretch from a centre), and Inversion, first single and then in various combinations. The simple operations are denoted by their capital initials thus, T, S, R, H, I. Compound operations are denoted by corresponding combinations of these in order from left to right: thus the point R obtained by the combination IS on a point P is the image in a given straight line of a point Q obtained from P by inversion from a centre. "Double points" play an important part throughout. Operations are always supposed to affect all points in the plane, and are treated algebraically by means of complex number, *e.g.* the above point P being denoted by $z=x+iy$, its inverse Q is $\dfrac{k^2}{x-iy}$. The image $R(z')$ of Q in Ox is therefore $\dfrac{k^2}{x+iy}$. Hence $zz'=k^2$. If the centre of inversion be taken at a point A whose abscissa is a, we have instead $(z-a)(z'-a)=k^2$. Similarly if the image is taken in the ordinate through A instead of $Ox (z-a)(z'-a)=-k^2$. Again let a, a' be abscissae of two points A, A' on Ox, the inversion I taken from A as centre, and the reflection S in the perpendicular bisector of AA', then $(z-a)(z'-a')+k^2=0$. It will illustrate the author's treatment if we apply this formula to the case of two successive inversions, from two different centres with different constants, considered in a note by Mr. R. F. Davis (*Math. Gazette*, vol. ii. p. 383), A and B being fixed points, P a variable point,

Q the inverse of P with centre A, constant k^2,

R „ „ „ Q „ „ B, „ k'^2,

so that R is obtained from P by a compound operation II'. It is required to show that this is equivalent to compound operation $I''S$, *i.e.* an inversion followed by a reflexion. Take the line through A, B for Ox and their abscissae as a, b; that z, z', z'' denote P, Q, R; then

$$(z-a)(z_0'-a)=k^2;\ (z''-b)(z_0'-b)=k'^2;$$

whence, eliminating z_0',

$$(b-a)(zz''-az''-bz+ab)-k^2(z''-b)+k'^2(z-a)=0,$$

which is equivalent to $(z-e)(z''-f)+l^2=0$, if

$$\left.\begin{aligned} e(b-a)&=k^2+a(b-a),\\ f(b-a)&=-k'^2+b(b-a),\\ (l^2+ef)(b-a)&=ab(b-a)+k^2b-k'^2a, \end{aligned}\right\}\quad \begin{aligned} (e-a)(b-a)&=k^2,\\ (b-f)(b-a)&=k'^2,\\ l^2&=(e-a)(b-f). \end{aligned}$$

Hence, if points E, F be taken on Ox with abscissae e, f,

$$AE \,.\, AB = k^2, \quad FB \,.\, AB = k'^2, \quad AE \,.\, FB = l^2,$$

and the result is seen to be an inversion I'' from E with constant $AE \,.\, FB$, followed by a reflexion S in the perpendicular bisector of EF, as proved geometrically by Mr. Davis.

Chapters I.-IV. deal with the transformations T, R, S, H and their combinations. V. with I, IS, SI. We note the omission of II' supplied above. VI. discovers the equations to groups of circles having two points in common (i) real, (ii) coincident, or (iii) imaginary. These are respectively

$$\text{(i)}\ \frac{z-a}{z-x'} = t\epsilon^{i\theta}; \quad \text{(ii)}\ \frac{1}{z} - \frac{1}{u} = t; \quad \text{(iii)}\ \frac{z-a}{z-a'} = k\epsilon^{-it}.$$

In all these t is a real quantity which may have any assigned value. The particular circle of the group considered is fixed in (i) by θ, in (ii) by the complex u, in (iii) by k; t giving by its different values different points on that circle.

VII. discusses the general cyclic transformation

$$z' = \frac{az+l}{a'z+l'}, \quad al' - a'l \neq 0.$$

VIII. applies previous results to the integration of a special system of differential equations

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

The work forms an excellent complement to Petersen's *Méthodes et Théories pour la résolution des Problèmes de constructions géométriques*. Each supplies something not given in the other—one the application of complex number, the other a large collection of problems for exercise.

Geometry, Practical and Theoretical. Vol. III. **Solid Geometry.** By V. Le Neve Foster. 3s. 6d. 1921. (G. Bell & Sons.)

"My object is to present elementary features of Solid Geometry *with many ideas as to their application*, in the hope that the practice will help to find other uses for the truths which have been known so long and yet are always fresh. The subject is developed both from the practical and theoretical point of view. *Reference is made continually to the historical side.* In the specimen examples the tendency has been to note how the ground should be prepared for the solution of the problem, by this means emphasising the necessary lines of attack." The author seems to have carried out this aim as stated above, especially as to the first of the passages we have italicised, rendering accessible to the student stores of information for which he might have to search in many different works, and not always find in a form suitable for his digestion. We should have been glad to see more and fuller historical references, but the limitations of available space may have made such extensions impossible. The propositions which he has demonstrated formally as fundamental are ten in number, ranging from the properties of parallelepipeds to Euler's Theorem $F + V = E + 2$. An appendix supplies proofs of five others which have been assumed, when required, as sufficiently obvious, but which some teachers may prefer to have demonstrated. There are five chapters numbered consecutively from the I.-XXXIX. in the previous volumes on Plane Geometry.

XL. gives propositions 38-40 on Parallelepipeds and Tetrahedra, followed (i) by numerical examples with diagrams on lines drawn on and through a cube, some fully worked out and others which leave calculation to the student, (ii) by a group of riders. XLI. gives props. 41-44 on Lines and Planes, followed by numerical and algebraical examples and riders. XLII. deals with Gradients. XLIII. with the Regular Solids and the interesting 'semi-regular' Rhombic Twelve-Face (see Mr. Hayward's paper, pp. 73-77 in vol. i.

of the *Mathematical Gazette*). XLIV. deals with Spheres. XLV.-XLVII. with areas and volumes connected with the bodies already discussed. XLVIII. is on Plans and Elevations. XLIX. gives prop. 46 on Solid Angles; L. prop. 47, Euler's. The examples here and exercises are specially good. LI., under the title 'The Earth,' contains an introduction to the study of Maps and Charts, pointing out difficulties which present themselves when these have to represent large portions of the surface. A page each is allowed to representations of 'Mercator's' and the 'Gnomonic' projections. These show, by the varied shapes and sizes on the maps of areas representing circles on the sphere, the general nature of the distortions produced. On each, courses by Rhumb Line and by Great Circle are traced for comparison. The construction of the meridians, parallels in a 'Zenithal Equal Area' projection, would form a useful exercise for the student on the theorems of Archimedes given on p. 527. The diagrams are numerous and excellent. Two minor criticisms occur to us.

(i) The Theorem underlying the construction of the perpendicular from a point to a plane—the Theorem of the Three Perpendiculars—should have been definitely enunciated and in more than one form, in view of its important applications in Practical Solid, some of which should have been pointed out.

(ii) Under 'Plans and Elevations' some of the simpler constructions of Lines and Planes, not necessarily with the notation of Descriptive Geometry, should be given, at least as exercises.

Calul des Erreurs Absolues et des Erreurs Relatives. By W. DE TANNENBERG. Pp. 35. n.p. 1922. (Vuibert.)

This is a useful brochure of 34 pages, about two-thirds of it being devoted to the consideration of the *absolute*, and the remainder to that of the *relative* errors possible when calculations are based on numerical constants only approximately known. The author explains carefully, with the aid of geometrical representations, the method adopted in reducing the number of decimal figures in a given decimal result exact or itself only approximately known. He uses formulae identical with those which deal with infinitesimals. Many examples are worked out. These generally deal with the evaluation of such expressions as $\pi\sqrt{5}$, $\frac{\sqrt{6}}{\pi}$, $\sqrt{\pi}$, where the fundamental constants can be obtained to any required degree of accuracy. We should have been glad if he had inserted a few more where the constants have been obtained through physical investigation, *e.g.* the mechanical equivalent of heat, in which the amount of accuracy is limited by the present state of arts and sciences. The author may of course fairly claim that the principles he lays down apply to such cases as well as to those he has considered, but the examination of a few of them might have served a useful purpose in impressing on some computers the fact, sometimes lost sight of, that the result of their calculations cannot, except by accident, have a higher degree of accuracy than the experimental data on which they are founded. EDWARD M. LANGLEY.

Cours Complet de Mathématiques Spéciales. Tome III. **Mécanique.** By J. HAAG. Pp. 188. 16 fr. 1922. (Paris, Gauthier-Villars.)

For general readers, this course falls between two stools. It does not begin at the beginning: the definitions of velocity and acceleration are dismissed as *considérations classiques élémentaires*, and stability is treated before equilibrium; and it does not go far enough for a second course: there is a very slight chapter on rigid dynamics, for which the author apologises in the preface.

There is a clear discussion of relative motion and relative force, attributed to Painlevé, which makes short work of the Parallelogram. The short chapter on units is good. Statics are treated as a particular case of dynamics, and most of the space allotted to it deals with machines in a conventional way.

There are no references except to the earlier volumes of the work.

H. P. H.

A Theory of Natural Philosophy. By ROGER JOSEPH BOSCOVICH, S.J. Latin-English edition, from the text of the first Venetian edition, published under the personal superintendence of the author in 1763, with a short Life of Boscovich. Pp. xix+470. £3 3s. 1922. (The Open Court Company, 149 Strand, W.C.)

This stately volume is a noble tribute to the memory of one of the members of a constellation which included Copernicus, Lobachevski, and Mendeljev. The government of the Kingdom of Serbs, Croats and Slovenes, with a few high-spirited Jugo-Slavs, have defrayed the expenses of publishing this, the sixth edition of the *magnum opus* of their great compatriot. He was born at Ragusa in 1711, fifteen years before the death of Newton. On his father's side he was of purely Serbian origin, and of this he was always proud. He owed his early training to the Jesuits, of whose Order he in due course became a member. Further, to fix the ideas, we may add that he became a Fellow of our Royal Society in 1761, in the same year as Joshua Reynolds, Erasmus Darwin, the grandfather of Charles, Robert Adam, the architect (who, with his brothers John, James and William, built the Adelphi, shortly to disappear), John Dollond, who invented the achromatic telescope and the modern heliometer, and Washington Shirley, the Earl Ferrers who made observations on the transit of Venus. He had some right to take his place among such men. We are reminded in this volume that he suggested or used a telescope filled with liquid for the purpose of measuring the aberration of light; at the same time as Maskelyne and Rochon, he invented a prismatic micrometer. He gave methods for determining the orbit of a comet from three observations, and the equator of the sun from three observations of a "spot." He attempted an explanation of the rings and investigated the orbit of Saturn. He invented a variable angle prism for measuring the refraction and dispersion of various kinds of glass. He successfully attacked the question of the earth's density and perfected the apparatus for, and advanced the theory of the measurement of the meridian. To G. de Saint-Jacques de Silvabelle is generally attributed the first satisfactory treatment of the problem of the solid of maximum attraction. But Boscovich, a year earlier, had dealt with the problem quite generally for a given mass, and for a law of attraction $f(r)$. He considered the logarithms of negative numbers, and before Bolyai and Lobachevski he had his doubts about the parallel postulate. Most of us associate his name with the auxiliary circle, and pp. lxxiii-lxxvii of Charles Taylor's *Ancient and Modern Geometry of Conics* will remind us how Boscovich guided his footsteps by the Baconian adage—"nihil natura facit per saltum"—and pushed home even to tyros the doctrine of continuity.

In the *Life* prefixed to this volume Prof. Petronievic claims for Boscovich that he was at once, philosopher, astronomer, physicist, mathematician, historian, engineer, architect, poet, diplomatist and man of the world.

Poet! well the Royal Society knew it. To them he dedicated his *De Solis et Lunae defectibus*, a Latin poem of some 5000 verses, with notes. And what is more, he found a doughty publisher, or pair of publishers, Millar and Dodsley, and a translator into French, the Abbé Baruel. Diplomatist he had need to be, for in few European scientific circles was a Jesuit a *persona grata*. D'Alembert hated him like poison, but Lalande, who travelled with him through Italy, was a Pythias to his Damon. Architect and astronomer—he made the plans for the observatory of Brera, of which he was for a time Director. Philosopher—metaphysical theorist—is not this in ink of red and black the title page of the *magnum opus*:

Theoria | Philosophiae Naturalis | redacta ad unicam legem virium | in natura existentium, | auctore | P. Rogerio Josepho Boscovich, | Societatis Jesu, | nunc ab ipso perpolita, et aucta, | Ac a plurimis praecedentium editionum | mendis expurgata. | Editio Veneta Prima | ipso auctore praesente, et corrigente. | Venetiis, | MDCCXLIII. | Ex typographia Remondiniana. | Superiorum permissu, ac privilegio. |

We can imagine Fra Gio. Paolo Zapparella, Inquisitor General of the Holy Office in Venice, uttering a sigh of relief as he completed his examination of the MSS. before testifying to the Censors of the College of Padua that the book contained nothing contrary to the Holy Catholic Faith. Be that as it

may, on September 18th, 1758, the book was registered in High Court for the Prevention of Blasphemy. To one unfamiliar with the history of the word paradox it would seem almost blasphemy to find the *Theoria* in the famous Budget. The theory according to which matter consists not of solid particles, but of mere mathematical centres of force, was as attractive to Faraday as it was to its author. Like Boscovich, Faraday displaced the atom from its proud pre-eminence and replaced it by a centre of force. Our Jesuit counted each body as composed of a number of geometrical points, from which, in obedience to certain mathematical laws, emanate forces, at certain small distances attractive, at other distances repulsive, and at greater distances attractive again. The cohesion of the parts of a body arises from these forces at the points, and to them we can also ascribe the resistance it exerts against the pressure of another body, and the attraction of gravitation which it exerts upon another body at a distance. Whewell saw in this, at least, a homogeneous and consistent theory, but he could not derive from it an explanation of the mechanical properties of a body as a whole, especially of its inertia. Many years later we find Sir William Thomson, as he then was, enlivening the journey between London and Glasgow by finding a complete settlement of the Boscovich theory of elastic solid. A homogeneous system of single points in equilateral tetrahedral order, each attracting or repelling its next neighbour, according as the distance between them is greater than, or is less than the edge of the static tetrahedron, gives a stable elastic solid. And in 1889 he was discussing the theory at the British Association Newcastle meeting. In 1895, Gray and he are making graphic histories of twenty-two encounters between a free particle (Boscovich) and a vibrator. In his preface to the English translation of Heinrich Herz's *Electric Waves*, Kelvin is almost lyrical. "Very soon after the middle of the eighteenth century, Father Boscovich gave his brilliant doctrine (if infinitely improbable theory), that elastic rigidity of solids, the elasticity of compressible liquids and gases, the attractions of chemical affinity and cohesion, the forces of electricity and magnetism; in short, all the properties of matter except heat, which he attributed to a sulphureous essence, are to be explained by mutual attractions and repulsions, varying solely with distances, between mathematical points endowed also, each of them, with inertia. Before the end of the eighteenth century the idea of action-at-a-distance through absolute vacuum had become so firmly established, and Boscovich's theory so unqualifiedly accepted as a reality, that the idea of gravitational force or electric force or magnetic force being propagated through and by a medium, seemed as wild to the mathematicians and naturalists of one hundred years ago as action-at-a-distance had seemed to Newton and his contemporaries one hundred years earlier. But a retrogression from the eighteenth-century school of science set in early in the nineteenth century." It was Faraday's discovery of specific inductive capacity that gave a new turn to the views of the younger generation. It will not, perhaps, be surprising that some of the ideas of Boscovich had been floating about before he gave them expression. A certain George Horne, five years before the first (Vienna) edition of the *Theoria*, quotes from a memoir by a Lincolnshire rector. "Mr. Rowning . . . has a very pretty conceit upon this same subject of attraction, about every particle of a fluid being intrenched in three spheres of attraction and repulsion, one within another, 'the innermost of which (he says) is a sphere of repulsion, which keeps them from approaching into contact; the next, a sphere of attraction, diffused around this sphere of repulsion, by which the particles are disposed to run together into drops; and the outermost of all, a sphere of repulsion, whereby they repel each other, when removed out of the attraction.' So that between the *urgings* and *solicitations* of the one and t'other, a poor unhappy particle must ever be at his wit's end, not knowing which way to turn or whom to obey first." Rowning, says De Morgan, has here started the notion which Boscovich afterwards developed. We also learn from Priestley that Boscovich was anticipated on another point, again by a country parson, John Michell, F.R.S., rector of Thornhill, near Halifax—also, by the way an architect, for he built, or had much to do with the building of, the "mathematical bridge" over the Cam at Queens'. Priestley, describing the apparatus used by Michell in his attempt to discover the momentum of light, states that Michell was

inclined to draw from his observations certain conclusions as to the "mutual penetrability of matter." He then observes that the ingenious hypothesis of Boscovich on the subject, or at least one the same in all essentials, occurred "to my friend Michell, in a very early period of his life, without his having had any communication with M. Boscovich, or even knowing that there was such a person. These two philosophers had even hit upon the same instances, to confirm and illustrate their hypotheses, especially those relating to contact, light and colour." Whatever put the idea into the head of Boscovich, it was not likely to be Baxter's *Immateriality of the Soul.* Michell found that Baxter imagined matter to consist, as it were, of bricks, cemented by immaterial mortar. Each brick was composed of smaller bricks, also cemented by immaterial mortar, and so, like the great fleas and lesser, *ad infinitum.* The rest of the story is to be found in pp. 81-82 of Sir Archibald Geikie's little monograph on Michell.

The Boscovichean classic is now presented to us on pages about fifteen inches by eleven, with what in these days we must call "glorious" margins, on which you may write in ink without fear of disaster. The Latin is on one page, and the English is opposite. The burden and heat of the work has fallen to the lot of Mr. J. M. Child, of Manchester University, and he has been fortunate to secure, for revision purposes, the assistance of Mr. A. O. Prickard, Fellow of New College, Oxford. The translators had no easy task, for they had to deal with a corrupt and badly punctuated text. But the result is a clear and readable translation, which we have judiciously sampled and adjudge a creditable performance of an always thankless task. One word of praise must be added for the way in which the printers, Messrs. Tanner and Butler, of Frome, have performed their share in the production of this fine volume.

First Course in the Theory of Equations. By L. E. Dickson. Pp. vi+168. 8s. 6d. net. 1922. (Messrs. Chapman, Hall.)

Dr. Dickson's little volume is the work of a skilled teacher. It must not be taken as a mere selection of sections from his *Elementary Theory of Equations.* "The proofs are simpler and more detailed, the exercises are simpler, more numerous, of greater variety, and involve more practical applications." He gives almost at once, as the fundamental theorem in Algebra, every algebraic equation with complex coefficients has a complex (real or imaginary) root, deferring the proof to the Appendix. He gives two methods for finding upper limits to the real roots of equations with real coefficients. Students and teachers alike will view with interest, Chapter III., on Ruler and Compass Construction. Here we have laid down as a criterion for constructibility:

> "A proposed construction is possible by ruler and compass if and only if the numbers which define analytically the desired geometric elements can be derived from those defining the given elements by a finite number of rational operations and extractions of real square roots."

With this and the theorem:

> "It is not possible with ruler and compass to construct a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root,"

the student is able to settle the thorny trisection of any angle and attack the problems of the construction of 9, 7, and 17-sided polygons.

A really excellent chapter on graphs brings us to Chapter VI. and Descartes' Rule of Signs. An elementary proof by Prof. D. R. Curtiss, here given, is worthy of attention. The Rule itself is later deduced as a corollary to Budan's Theorem. Chapter VII. would delight the heart of De Morgan. We quote from the preface:

> "Here is developed a method of computing a real root of an equation with minimum labour and with certainty as to the accuracy of all the decimals obtained. We first find by Horner's method successive transformed equations whose number is half of the desired number of significant figures of the root. The final equation is reduced to a linear equation by applying to the constant term the correction computed from the omitted terms of the second and higher degrees, and the work is completed by abridged division. The method combines speed with control of accuracy."

Newton's method is discussed both graphically and analytically, and we are given examples of its applications to functions not polynomials. The twenty-five pages on determinants are followed by chapters on Symmetric Functions, Elimination, Resultants and Discriminants. The author is to be congratulated on an exceptionally sound and useful piece of work.

James Stirling. A Sketch of the Life and Works along with his Scientific Correspondence. By C. Tweedie. Pp. x+213. 16s. net. 1922. (Clarendon Press.)

The interesting account of James Stirling, the Venetian, contributed to the *Gazette* (x. p. 119) by Mr. Tweedie will but have whetted the appetite for this sketch of the life and work, with the scientific correspondence, of the author's famous fellow-countryman. The letters here given are between Stirling and Maclaurin, Cramer, Nicolas Bernoulli, L'Abbé Castel, Campailla, J. Bradley, Klingenstierna, Machin, Clairaut, Euler, and Martin Folkes. We find many points of interest in the letters apart from their strictly mathematical content.

Thus Maclaurin recommends to Stirling a "modest, industrious and accurate young man," named Maitland, as having a natural turn for making mathematical instruments. He seems to have been satisfactory, but apparently did not become one of the great princes of the craft, inasmuch as his name does not appear on Mr. Gunther's list. We are also told that owing to the scandalous behaviour of "the French Mathematicians in Peru, the natives murdered their servants, destroyed their Instruments, and burn't their papers, the Gentlemen escaping narrowly themselves. What an ugly Article will this make in a Journal." Maclaurin anxiously enquires (1734), if there is any truth in the newspaper reports that Halley has discovered the longitude. We find him sending to Stirling a copy of a letter from Maupertuis to Bradley, announcing the confirmation of Newton's view that the earth is an oblate spheroid. This seems to have upset Cassini, and Maclaurin chuckles over the current gossip: "I am told that Mr. Cassini would willingly find some fault with the Observations to save his father's doctrine, but is so much at a loss that he is obliged to suppose that the instrument was twice disordered."

Eventually a certificate had to be produced from England to verify the claim that the instrument used (Graham's) could not be "turned contrarywise." Mr. Tweedie's volume is not only a very useful, but an attractive piece of work. Before closing this notice we pass on a couple of queries, (1) Cramer opens a letter thus: Viro Clarissimo, Doctissimo Jacobo Stirling, L.A.M. & R.S. Socio, Gabriel Cramer, S.P.D. What is the meaning of L.A.M.? (2) Did Goldbach ever meet Newton?

Early Science in Oxford. Part II.—**Mathematics.** By R. T. Gunther. Pp. 101. 10s. 6d. net. 1922. (Oxford University Press.)

The first part of these monographs was published a couple of years ago, and dealt with Chemistry. The third part is to deal with Astronomy, and await, we are told, the collection of a guarantee fund to cover the cost of the plates. Part II. consists of three sections—Notes on Early Mathematicians: Early Mathematical Instruments belonging to the University and Colleges of Oxford: and Notes on Mathematical Instrument Makers. After short accounts of Daniel of Morley, John of Holywood, and John of Basingstoke, and an exposition of what mathematical science really meant in the England of the eleventh to the thirteenth centuries, the author brings us to the splendid group—Richard of Wallingford, Maudith, Simon Bredon, Ashenden, Rede, etc. Their work and numbers entitle him to claim that in the fourteenth century, Oxford could boast more Mathematicians than any country in Europe. Some day, when we have Chairs for the study of the History of Science, it will occur to some one that the manuscripts of men like these are worth re-examination, and perhaps even publication. As Mr. Gunther reminds us, so little was known of Bradwardine by professed historians that the *Doctor Profundus* was placed by Montucla at the beginning of the sixteenth century. And he laments that "for the decyphering of papyri and for any ramification of archaeological research, helpers and money have been readily forthcoming, but records of the scientific work of scholars of a period in which English science was pre-eminent are still awaiting an editor."

Readers will follow the author closely in his very instructive and entertaining account of the Oxford mathematicians. The second section is really much more than a descriptive catalogue of Oxford mathematical instruments, for it is enriched with a comparative account of many other collections. Not

merely do we get careful descriptions of the instruments, but they are accompanied by chronological and biographical material of the greatest interest. The final section, consisting of notes on makers of mathematical instruments, is a pious and happy fulfilment by a modern librarian of one of the duties of his craft as laid down by John Pell: "To keep a Catalogue of all such workmen as are able and fit to bee imploied in making Mathematical Instruments and representations, working upon Wood, Magnets, Metals, Glass, etc." Mr. Gunther must be cordially congratulated on this beautifully illustrated and most welcome contribution to a field of research that has been strangely neglected.

SYDNEY BRANCH.

Report for the Year 1922.

The members now number 24, and the associates 40: total 64. The first meeting held this year was addressed by the President on "The Present Position of Mathematics in England"—the substance of the address, which was greatly appreciated, being impressions gained by Prof. Carslaw during his sojourn of more than a year in the United Kingdom. In the second term, a well-attended meeting was held to consider the "new draft rules for the Leaving Certificate examination; and the contents of the Mathematics syllabus for each paper." In the third term, the Association was addressed by Mr. W. F. Gale, F.R.A.S., President of the Royal Astronomical Society of Australia, on "Some Solar Phenomena; with special reference to the recent eclipse." The address was illustrated throughout by lantern slides, and was greatly appreciated. At the same meeting, Mr. J. Nangle, F.R.A.S., also spoke about observational work carried out at Stanthorpe.

The office-bearers appointed for 1923 were: *President,* Prof. Carslaw; *Hon. Treasurer,* Mr. C. A. Gale; *Joint Hon. Secretaries,* Miss Janet Brown, Mr. H. J. Meldrum. Miss Cohen's resignation from the position of Joint Hon. Sec. was received with regret; Miss Cohen was thanked for her past services to the Association.

H. J. Meldrum (*Hon. Sec.*).

YORKSHIRE BRANCH.

A meeting of the Yorkshire Branch of the Mathematical Association was held on Saturday, Feb. 17, at University House, Leeds. Miss Stephen, Senior Mathematical Mistress at the Leeds Girls' High School, was elected Treasurer in place of Mrs. Pochin, who left Leeds at Christmas to undertake the duties of a Headmistress-ship. It was decided to use a half of the balance of the funds of the Branch in purchasing a book to be presented to Leeds University, and a donation of £5 to the London Mathematical Society towards the publishing of original research work. Miss Stephen then gave an account of her "Tour in America" and of her impressions of American Education—Mathematical and general—in the course of which she made some comparisons between the high value placed on, and the readiness to pay for education there and the attitude of many in this country. Mr. Sewell, of Wakefield Grammar School, then gave a paper on Elementary Mathematics and Geography, in which he showed how Geometry could be utilised in and illustrated by simple geography.

NORTH WALES BRANCH.

The N. Wales Branch held no meetings between 12th Nov., 1921, and 25th Nov., 1922—the deaths of Prof. Mathews and Mr. Creak having somewhat disorganised the Branch's activities. At a meeting on 25th November it was decided to continue the Branch under the original rules, and the following officers were elected: *President,* Prof. G. H. Bryan, Sc.D., F.R.S.; *Vice-President,* Mr. W. F. Ferrar, M.A.; *Secretary,* Mr. E. R. Hamilton, M.A., B.Sc.

Three meetings were arranged for the rest of the present school year 27th Jan.: "The Application of Statistics to Educational Problems," by Mr. E. R. Hamilton; 28th Mar.: "The Teaching of Geometry," by Mr. A. P. Chepman. Summer Term (date not fixed). A meeting to discuss the statistical inquiries suggested at the first meeting.

E. R. Hamilton, *Secretary.*

ERRATA.

P. 24, l. 9 up. *For* "0·919" *read* "9·319."

P. 197, last line. *For* "$\beta \cos C = \gamma \cos B$" *read* "$b^2\beta \cos C = c^2\gamma \cos \beta$."

P. 198, l. 16. *For* "$\beta \cos C = -\gamma \cos B$" *read* "$b^2\beta \cos C = c^2\gamma \cos C$."

[The point at which the two loci in question cut BC is the foot of the perpendicular from A_1.]

Obituary.

RAWDON LEVETT.

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THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"*I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.*"—BACON.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Southampton, Bangor, and Sydney (New South Wales). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) SHORT NOTICES of books not specially dealt with in the REVIEWS;

(5) QUERIES AND ANSWERS, on mathematical topics of a general character.